ON GENERALIZED QUATERNION ALGEBRAS

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ABSTRACT. Let B be a commutative ring with 1, and G (={o}) an automorphism group of B of order 2. The generalized quaternion ring extension $B[j]$ over $B$ is defined by S. Parimala and R. Sridharan such that (1) $B[j]$ is a free $B$-module with a basis $\{1,j\}$, and (2) $j^2 = -1$ and $j b = o(b) j$ for each $b$ in $B$. The purpose of this paper is to study the separability of $B[j]$. The separable extension of $B[j]$ over $B$ is characterized in terms of the trace (= $1 + o$) of $B$ over the subring of fixed elements under $o$. Also, the characterization of a Galois extension of a commutative ring given by Parimala and Sridharan is improved.

KEY WORDS AND PHRASES. Quaternion Rings, Separable Algebras, and Galois Extensions.

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1. INTRODUCTION.

In [6], we studied the separable extension of group rings $RG$ and quaternion rings $R[i,j,k]$ over a ring $R$ with $1$. We have shown that $R[i,j,k]$ is a separable extension of $R$ if and only if $2$ is a unit in $R$. Recently, S. Parimala and R. Sridharan ([5]) investigated another class of quaternion ring extensions $B[j]$ over a commutative ring $B$ with $1$ and with an automorphism group $G (= \{6\})$ of order $2$, where $B[j]$ is a free $B$-module with a basis $\{1, j\}$, $j^2 = -1$, and $jb = 6(b)j$ for each $b$ in $B$. Their work is based on the following characterization of a Galois extension of a commutative ring ([5], Proposition 1.1): Let $A$ be the set of elements in $B$ fixed under $\sigma$. Assume $2$ is a unit in $A$. Then, $B$ is Galois over $A$ if and only if $B \otimes_A B[j] \cong M_2(B)$, a matrix algebra over $B$ of order $2$, where the Galois extension is in the sense of Chase-Harrison-Rosenberg ([2]). The purpose of this paper is to study the separability of $B[j]$. Without the assumption that $2$ is a unit in $A$, we shall characterize the separability of $B[j]$ in terms of the trace ($= 1 + 6$) of $B$ over $A$. This shows the existence of a separable generalized quaternion ring extension $B[j]$ with $2$ not a unit in $A$. When $\text{Char}(A) = 2$, we shall show that $B[j]$ is a separable extension over $B$ if and only if $B$ is Galois over $A$. Thus we can improve the above theorem of Parimala and Sridharan. Then, the case in which $2$ is a unit will be discussed, and several examples are constructed to illustrate our main results.

2. PRELIMINARIES.

Let us recall some basic definitions as given in [1], [2], [3], [4] and [6]. Let $B$ be a commutative ring containing a subring $A$ with the same identity $1$. Then $B$ is called a Galois extension over $A$ ([2], or [3], Chapter 3) with a finite automorphism group $G$ if (1) there exist
elements \{a_i, b_i \in B / i = 1, 2, \ldots, n\} such that
\[ \sum a_i b_i = 1 \text{ and } \sum a_i \sigma(b_i) = 0 \text{ whenever } \sigma \neq 1 \text{ in } G, \] and (2) \( A = \{ b \in B / \sigma(b) = b \text{ for all } \sigma \in G \}. \] The map \( \sum \sigma \) is called the trace of \( B \) over \( A \) denoted by \( \text{Tr} \). Let \( S \) be a ring (not necessarily commutative) containing a subring \( R \) with the same identity 1. Then \( S \) is called a separable extension of \( R \) if there exist elements, \{c_i, d_i \in S / i = 1, 2, \ldots, n\} such that (1) \( a(\sum c_i d_i) = (\sum c_i d_i)a \) for all \( a \in S \) where \( \sum \) is over \( R \), and (2) \( \sum c_i d_i = 1 \). Such an element \( \sum c_i d_i \) is called a separable idempotent for \( S \). When \( R \) is contained in the center of \( S \), \( S \) is called a separable \( R \)-algebra. The separable \( R \)-algebra \( S \) is called an Azumaya \( R \)-algebra if \( R \) is the center of \( S \).

3. SEPARABLE QUATERNION ALGEBRAS.

Throughout, we assume that \( B \) is a commutative ring with 1, and \( G (= \{ \sigma \} ) \) an automorphism group of order 2 of \( B \), and that \( B[j] \) is the generalized quaternion algebra over \( A \), where \( A \) is the subring of elements fixed under \( \sigma \). Our main goal in the section is to study a separable extension \( B[j] \) over \( B \) without the assumption that 2 is a unit in \( A \). We begin with a description of the set of separable idempotents for \( B[j] \) (if there are any) over \( B \). Clearly, \( \{1j, 1j, j, j \} \) is a basis for \( B[j] \).

LEMMA 3.1. The element \( x = a_{11}(1j) + a_{22}(1j) + a_{21}(j) + a_{12}(j) \) is a separable idempotent for \( B[j] \) over \( B \) if and only if (1) \( a_{22} = -\sigma(a_{11}) \) such that \( \text{Tr}(a_{11}) = 1 \), and (2) \( a_{21} = \sigma(a_{12}) \) such that \( a_{12}((b - \sigma(b)) = 0 \) for all \( b \) in \( B \) and \( \text{Tr}(a_{12}) = 0 \).

PROOF. Let \( x \) be a separable idempotent for \( B[j] \) over \( B \). Then \( xu = ux \) for each \( u \) in \( B[j] \). Hence \( xj = jx \); that is,
\[ \sigma(a_{11})(j) + \sigma(a_{12})(j) - \sigma(a_{21})(1j) - \sigma(a_{22})(1j) = \]
Equating corresponding coefficients, we have $a_{11} = -a_{22}$, $a_{12} = a_{21}$; that is, $a_{22} = -6(a_{11})$ and $a_{21} = 6(a_{12})$ for $6^2 = 1$. Also, $bx = xb$ for all $b$ in $B$, so $b_{12}(b - 6(b)) = 0$. Thus $x = a_{11}(101) + a_{12}(10j) + 6(a_{12})(j0j) - 6(a_{11})(j0j)$ with $a_{12}(b - 6(b)) = 0$. Moreover, by the second condition of a separable idempotent, $a_{11} + (a_{12} + 6(a_{12}))j + 6(a_{11}) = 1$, so $Tr(a_{11}) = 1$ and $Tr(a_{12}) = 0$. Conversely, it is straightforward to verify that any $x$ satisfying all equations as given is a separable idempotent.

**Theorem 3.2.** $B[j]$ is a separable extension over $B$ if and only if there is an element $c$ in $B$ such that $Tr(c) = 1$.

**Proof.** The necessity is a consequence of Lemma 3.1. For the sufficiency, if $Tr(c) = 1$, we take $a_{11} = c$, $a_{12} = a_{21} = 0$. Then $a_{11}(101) - 6(a_{11})(j0j)$ is a separable idempotent for $B[j]$ by Lemma 3.1. Thus $B[j]$ is a separable extension over $B$.

Using Theorem 3.2, we can obtain a characterization of a separable extension $B[j]$ over $B$ when $Char(A) = 2$.

**Theorem 3.3.** Assume $Char(A) = 2$. Then, $B[j]$ is a separable extension over $B$ if and only if $B$ is a Galois extension over $A$.

**Proof.** Let $B$ be a Galois extension over $A$. Corollary 1.3 on P. 85 in [3] implies that $Tr(c) = 1$ for some $c$ in $B$. Thus $B[j]$ is a separable extension over $B$ by Theorem 3.2. Conversely, by Theorem 3.2 again, there exists an $c$ in $B$ such that $Tr(c) = 1$, so $(c+6(c)) = 1$. By hypothesis, $Char(A) = 2$, $6(c) = 6(-c) = -6(c)$, so $c - 6(c) = 1$. Hence the ideal generated by $\{(b - 6(b)) / b \in B\} = B$. This implies that $B$ is Galois over $A$ by the statement 5 in Proposition 1.2 on P. 81 in [3].

Let us recall that the theorem of Parimala and Sridharan (Proposition 1.1 in [5]): Assume $2$ is a unit in $A$. Then, $B$ is Galois over $A$. 


if and only if $B^q_A B[j] \cong M_2(B)$, a matrix algebra over $B$ of order 2.

We are going to improve it without the assumption that 2 is a unit in $A$.

**THEOREM 3.4.** If $B$ is Galois over $A$, then $B^q_A B[j] \cong M_2(B)$.

**PROOF.** If $B$ is Galois over $A$, there exists an $c$ in $B$ such that

$$\text{Tr}(c) = 1$$

([3], Corollary 1.3, P. 85). Hence $B[j]$ is a separable extension over $A$ by Theorem 3.2. But $B$ is also a separable extension over $A$ by Proposition 1.2 in [3], so the transitive property of separable extensions ([4], Proposition 2.5) implies that $B[j]$ is a separable $A$-algebra. Moreover, we claim that (1) $B[j]$ is an Azumaya algebra over $A$, and (2) $B$ is a maximal commutative subalgebra of $B[j]$. The proof of these facts was given in [7]. For completeness, we give an outline here. For part (1), it suffices to show that $A$ is the center of $B[j]$. Clearly, $A$ is contained in the center. Now, let $b+b'j$ be in the center. Then $j(b+b'j) = (b+b'j)j$ and $c(b+b'j) = (b+b'j)c$ for each $c$ in $B$. Equating coefficients of the basis $\{1,j\}$ in the above equations, we have that $b$ is in $A$ and $b' = 0$ by Statement 5 in Proposition 1.2 on P. 81 in [3]. For part (2), to show that $B$ is a maximal commutative subalgebra of $B[j]$ is to show that the commutant of $B$ in $B[j]$ is $B$.

The computation is similar to part (1).

Moreover, noting that $B$ is separable over $A$, we then conclude

$$B^q_A (B[j])^O \cong \text{Hom}_B(B [j], B [j])$$

by Theorem 5.5 on P. 65 in [3], and this implies that $B^q_A B[j] \not\cong M_2(B)$, where $(B[j])^O$ is the opposite ring.

In [7], the sufficiency of the Parimala and Sridharan theorem was shown by a different method from [5]. Now we slightly improve the statement without the assumption that 2 is a unit in $A$.

**THEOREM 3.5.** Let $B[j]$ be a separable extension over $B$. If $B^q_A B[j] \cong M_2(B)$, then $B$ is Galois over $A$. 


PROOF. Since $B[j]$ is a separable extension over $B$, there exists an element $c$ in $B$ such that $\text{Tr}(c) = 1$ by Theorem 3.2. Hence the sequence $B \to A \to 0$ is exact under the trace map. But $A$ is projective over $A$, so the sequence splits, and then $A$ is an $A$-direct summand of $B$. By hypothesis, $B \otimes_A B[j] \cong M_2(B)$ which is an Azumaya $B$-algebra, so $B[j]$ is an Azumaya $A$-algebra ([3], Corollary 1.10, P. 45). Therefore $B$ is Galois over $A$ by using the same argument as given in [7].

In Theorem 3.5, the hypothesis that $B \otimes_A B[j] \cong M_2(B)$ can be replaced by that $B \otimes_A B[j]$ is an Azumaya $B$-algebra with the same proof.

4. SPECIAL SEPARABLE QUATERNION ALGEBRAS.

Theorem 3.5 tells us that $B[j]$ is an Azumaya $A$-algebra such that $B \otimes_A B[j] \cong M_2(B)$ when $B$ is Galois over $A$. In this section, we are going to discuss generalized quaternion algebras $B[j]$ in which 2 is a unit in $A$ when $B$ is projective and separable over $A$. With a similar argument as given in Lemma 3.1, we have

LEMMA 4.1. The element $a_{11}(1o1)+a_{12}(1o1)+a_{21}(j01)+a_{22}(j0j)$ in $A[j] \otimes_A B[j]$ is a separable idempotent for $A[j]$ if and only if (1) $a_{22} = -a_{11}$ such that $2a_{11} = 1$, and (2) $a_{21} = a_{12}$ such that $2a_{12} = 0$.

THEOREM 4.2. The $A$-algebra $A[j]$ is separable if and only if 2 is a unit in $A$.

PROOF. The necessity is clear by Lemma 4.1; the sufficiency is immediate because $(1/2)(1o1-j0j)$ is a separable idempotent.

Now we give a characterization of $B[j]$ in which 2 is a unit when $B$ is projective and separable over $A$.

THEOREM 4.3. Let $B$ be separable and projective over $A$. Then, $B[j]$ is a separable extension over $B$ and projective over $A[j]$ as a bimodule if and only if 2 is a unit in $A$. 
PROOF. Let 2 be a unit in A and let c be \((1/2)\). Then \(\text{Tr}(c) = 1/2 + 1/2 = 1\), and hence \(B[j]\) is separable over \(B\) by Theorem 3.2. By hypothesis, \(B\) is projective over \(A\), so \(B[j]\) is left projective over \(A\) (for \(B[j]\) is left projective over \(B\)). Hence \(B[j]\) is left projective over \(A[j]\) ([3], Proposition 2.3, P. 48). We next claim that \(B[j]\) is also right projective over \(A[j]\). In fact, \(\alpha: B[j]A[j] \rightarrow B[j]\) defined by \(\alpha(b + b'j) = b\alpha b'j\) for all \(b\) and \(b'\) in \(B\) is an isomorphism as right \(A[j]\)-modules. But \(B\) is projective over \(A\), so \(B[j]A[j]\) is right projective over \(A[j]\). This proves that \(B[j]\) is right projective over \(A[j]\). Thus \(B[j]A[j] \rightarrow (B[j])^O\) is projective as \(A[j]-A[j]\)-module. Since \(B[j]\) is a direct summand of \(B[j]A[j] \rightarrow (B[j])^O\) as a \(B[j]A[j]\)-module (for \(B[j]\) is separable over \(A\)), \(B[j]\) is projective as a \(A[j]-A[j]\)-module.

Conversely, to show that 2 is a unit in \(A\), it suffices to show that \(A[j]\) is a separable \(A\)-algebra by Theorem 4.2. Since \(B[j]\) is a separable extension over \(B\), \(\text{Tr}(c) = 1\) for some \(c\) in \(B\) by Theorem 3.2. Hence \(\text{Tr}: B \rightarrow A \rightarrow 0\) is exact. We claim that \(\text{Tr}\) induces an exact sequence: \(B[j] \rightarrow A[j] \rightarrow 0\) as \(A[j]-A[j]\)-modules. We define \(\beta: B[j] \rightarrow A[j] \rightarrow 0\) by \(\beta(b + b'j) = \text{Tr}(b) + \text{Tr}(b')j\). Clearly, \(\beta\) is an additive group homomorphism. Moreover, for \(a, a'\) in \(A\), \((b + b'j)(a + a'j) = (ba + b'a') + (ba' + b'a)j\), so \(\beta((b + b'j)(a + a'j)) = \text{Tr}(ba - b'a') + \text{Tr}(ba' + b'a)j = (a\text{Tr}(b) - a'\text{Tr}(b')) + (a'\text{Tr}(b) + a\text{Tr}(b'))j\). Also, \(\beta((b + b'j)(a + a'j) = (\text{Tr}(b) + \text{Tr}(b')j)(a + a'j) = \beta((b + b'j)(a + a'j))\). Thus \(\beta\) is a right \(A[j]\)-homomorphism. Similarly, by noting that \(\text{Tr} = 1 + 6\) and that \((\text{Tr})6 = \text{Tr} = 6(\text{Tr})\), it is straightforward to verify that \(\beta\) is a left \(A[j]\)-homomorphism. But then \(A[j]\) is \(A[j]-A[j]\) projective such that \(\beta\) is onto (for \(\text{Tr}(c) = 1\) in \(A[j]\)). This implies that the exact

5. EXAMPLES.

This section includes several examples to illustrate our results.

(1) Let \( Z \) be the ring of integers, and \( Z \times Z (= B) \) the ring of direct product of \( Z \) under the componentwise operations. Define \( \sigma: Z \times Z \rightarrow Z \times Z \) by \( \sigma(a,a') = (a',a) \) for \( a,a' \) in \( Z \). Then \( \sigma \) is an automorphism group of order 2 and \( \{(a,a) / a \in Z\} (= A) \) is the subring of \( Z \times Z \) of the fixed elements under \( \sigma \). Imbed \( Z \) in \( Z \times Z \) by \( a \rightarrow (a,a) \). Then we have

(a) \( Z \times Z \) is a free \( A \)-module with a basis \( \{(1,0),(0,1)\} \).

(b) \( Z \times Z \) is separable over \( Z \).

(c) \( (Z \times Z)[j] \) is a separable extension over \( Z \times Z \) because \( \text{Tr}((1,0)) = (1,0)+(0,1) = (1,1) \) by Theorem 3.2.

(d) \( Z[j] \) is not separable over \( Z \) because 2 is not a unit in \( Z \) by Theorem 4.2.

(e) \( (Z \times Z)[j] \) is not projective over \( Z[j] \) because 2 is not a unit in \( Z \) by Theorem 4.3.

(2) Let \( Z(3) \) be the local ring of \( Z \) at the prime ideal \( (3) \). Replace \( Z \) by \( Z(3) \) in Example (1). Then we have

(a) 2 is a unit in \( Z(3) \).

(b) All properties (a), (b) and (c) in Example (1) hold.
(c) \((Z(3)^{xZ(3)})[3]\) is projective over \(Z(3)[3]\) by Theorem 4.3.

(3) \(Z \times Z\) and \(Z(3)^{xZ(3)}\) in Example (1) and Example (2) are Galois over \(Z\) and \(Z(3)^{xZ(3)}\) respectively by using Proposition 1.2 on p. 64 in [3],
Since \(\text{Tr}(3,-2)) = (3,-2)+(-2,3) = (1,1)\) which is not in any maximal ideal of \(Z \times Z\) or \(Z(3)^{xZ(3)}\). Thus \((Z \times Z)[Z \times Z][3] \cong M_2(Z \times Z)\) and \((Z(3)^{xZ(3)})[Z(3)^{xZ(3)}][3] \cong M_2(Z(3)^{xZ(3)}))\) by Theorem 3.4.

(4) Let \(i\) be the usual imaginary unit. Then \(Z[i]\) is not separable over \(Z\). \(Z[i]\) has an automorphism group \(\{\sigma: \sigma(a+bi) = a-bi\) for \(a, b\) in \(Z\}\) such that \(\sigma^2 = 1\) and \(Z\) is the fixed ring of \(\sigma\). Also, (a) \((Z[i])[i]\) is not separable over \(Z[i]\), and (b) \(Z[i]\) is not Galois over \(Z\).

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