A NOTE ON THE SUBCLASS ALGEBRA

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ABSTRACT. Each irreducible character of the subclass algebra is paired up with its irreducible module.

KEY WORDS AND PHRASES. Finite group, irreducible character, subclass algebra.

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INTRODUCTION.

Let G be a finite group and let H be a subgroup of G. If g ∈ G, the subclass of G containing g is the set \( E_g = \{ hgh^{-1} | h \in H \} \) and the subclass sum containing g is \( B_g = \sum_{x \in E_g} g \). The algebra over the complex numbers, K, generated by these subclass sums is called the subclass algebra (denoted by S) associated with G and H.

Let \( \{ M_1, \ldots, M_s \} \) be the irreducible KG-modules with \( M_j \) affording the irreducible character, \( \chi_j \), of G and let \( \{ N_1, \ldots, N_t \} \) be the irreducible KH-modules with \( N_i \)
affording the irreducible character $\phi_i$, of $H$. Suppose $\{e_i\}_{i=1}^t$ is a set of primitive orthogonal idempotents of $KH$ and $\{f_i\}_{i=1}^t$ is the set of primitive central orthogonal idempotents of $KH$ where the sets are indexed so that $N_i = KH e_i$ and $f_i = (\dim N_i) e_i = \frac{\dim \phi_i}{|H|} \sum_{h \in H} \phi_i(h^{-1}) h$. We define the non-negative integers $\{c_{ij}\}$ by

$$\chi_j \bigg|_H = \sum_{i=1}^t c_{ij} \phi_i.$$  

In [2], it was demonstrated that the irreducible $S$-modules are $\{e_i M_j\}$.

D. Travis [3] has shown that the irreducible characters of $S$ are parameterized by pairs $\chi_j, \phi_1 (c_{ij} \neq 0)$ and are given by

$$\psi_{ij}(g) = \frac{|E_g|}{|H|} \sum_{h \in H} \chi_j(gh) \phi_1(h^{-1}). \quad (1)$$

Independent of Travis's work we show that the irreducible character afforded by $e_i M_j$ is $\psi_{ij}$.

**LEMMA:** $\chi_i(sB_g) = |E_g| \chi(sg) \forall s \in S, \forall g \in G$

**PROOF:** Since $B_g = \frac{|E_g|}{|H|} \sum_{h \in H} ghg^{-1}$, we have $\chi_i(sB_g) = \frac{|E_g|}{|H|} \sum_{h \in H} \chi_i(ghg^{-1})$

$$= \frac{|E_g|}{|H|} \sum_{h \in H} \chi_i(hsg^{-1}) \text{ since } hs = sh, \forall h \in H$$

$$= |E_g| \chi_i(sg).$$

**THEOREM:** Let $\psi_{ij}$ be the character afforded by the irreducible $S$-module $e_i M_j (c_{ij} \neq 0)$. Then $\psi_{ij}$ is as defined by equation (1).

**PROOF:** By proposition 2.3 of [2], we have $M_j |^S = \sum_{k=1}^t (\dim N_k) e_k M_j$

$$= \sum_{k=1}^t \sum_{j} f_k M_j.$$

Therefore, for $s \in S$, $\chi_j(sf_i) = \sum_{k=1}^t \sum_{j} (\dim \phi_k) \psi_{kj}(sf_i)$.
\[(\dim \phi_i) \psi_{ij}(s f_i)\]
\[(\dim \phi_i) \psi_{ij}(s)\]

since the trace of the action of \(s f_i\) on \(f_i M_j\) is the same as the trace of the action of \(s\) on \(f_i M_j\) and the trace of the action of \(s f_i\) on \(f_k M_j\) \((i \neq k)\) is 0.

Thus \(\psi_{ij}(B_g) = \frac{1}{\dim \phi_i} \chi_j(B_g f_i)\)
\[= \frac{|E_g|}{\dim \phi_i} \chi_j(g f_i)\]
by the Lemma
\[= \frac{|E_g|}{|H|} \sum_{h \in H} \chi_j(gh) \Phi_i(h^{-1})\].

End of proof.

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REFERENCES

