GENERALIZED DERIVATION MODULO THE IDEAL OF ALL COMPACT OPERATORS

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We give some results concerning the orthogonality of the range and the kernel of a generalized derivation modulo the ideal of all compact operators.

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1. Introduction. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators acting on a complex Hilbert space $\mathcal{H}$. For $A$ and $B$ in $\mathcal{L}(\mathcal{H})$, let $\delta_{A,B}$ denote the operator on $\mathcal{L}(\mathcal{H})$ defined by $\delta_{A,B}(X) = AX - XB$. If $A = B$, then $\delta_A$ is called the inner derivation induced by $A$.

In [1, Theorem 1.7], Anderson showed that if $A$ is normal and commutes with $T$ then, for all $X \in \mathcal{L}(\mathcal{H})$,

$$ ||T - (AX -XA)|| \geq ||T||. \quad (1.1) $$

In [4], we generalized this inequality, we showed that if the pair $(A,B)$ has the Putnam-Fuglede’s property (in particular if $A$ and $B$ are normal operators) and $AT = TB$, then for all $X \in \mathcal{L}(\mathcal{H})$,

$$ ||T - (AX -XB)|| \geq ||T||. \quad (1.2) $$

The related inequality (1.1) was obtained by Maher [3, Theorem 3.2] who showed that, if $A$ is normal and $AT = TA$, where $T \in C_p$, then $||T - (AX -XA)||_p \geq ||T||_p$ for all $X \in \mathcal{L}(\mathcal{H})$, where $C_p$ is the von Neumann-Schatten class, $1 \leq p < \infty$, and $|| \cdot ||_p$ its norm. Here we show that Maher’s result is also true in the case where $C_p$ is replaced by $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators with $|| \cdot ||_\infty$ its norm. Which allows to generalize these results, we prove that if the pair $(A,B)$ has the Putnam-Fuglede’s property in $\mathcal{K}(\mathcal{H})$, and $AT = TB$, where $T \in \mathcal{K}(\mathcal{H})$, then $||T - (AX -XB)||_\infty \geq ||T||_\infty$ for all $X \in \mathcal{L}(\mathcal{H})$.

2. Normal derivations. In this section, we investigate on the orthogonality of the range and the kernel of a normal derivation modulo the ideal of all compact operators. We recall that the pair $(A,B)$ has the property $(\text{PF})_{\mathcal{K}(\mathcal{H})}$ if $AT = TB$, where $T \in \mathcal{K}(\mathcal{H})$ implies $A^*T = TB^*$. Before proving this result we need the following lemmas.

**Lemma 2.1.** Let $N, X \in \mathcal{L}(\mathcal{H})$, where $N$ is a diagonal operator. If $\delta_N(X) + S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and $||\delta_N(X) + S||_\infty \geq ||S||_\infty$. 
**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the diagonal operator $N$. Then, the operator $N$ can be written under the following matrix form:

$$
\begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_n
\end{bmatrix}.
$$

(2.1)

According to the following decomposition of $H$:

$$
H = \bigoplus_{i=1}^{n} \ker(N - \lambda_j).
$$

(2.2)

Let $|\delta_{ij}|$ and $|X_{ij}|$ be the matrix representations of $S$ and $X$ according to the above decomposition of $H$. Then

$$
NX - XN = |(\lambda_i - \lambda_j)X_{ij}|.
$$

(2.3)

Since $S \in \{N\}'$ (the commutant of $N$), we get $S_{ij} = 0$ for $i \neq j$. Consequently

$$
NX - XN + S = 
\begin{bmatrix}
S_{11} & * & * & * \\
* & S_{22} & * & * \\
* & * & * & * \\
* & * & S_{nn}
\end{bmatrix}.
$$

(2.4)

Here $*$ stands for some entry.

As $\delta_N(X) + S \in \mathcal{H}(H)$, so $S \in \mathcal{H}(H)$ and the result of Gohberg and Krein [2] guarantee that $\|\delta_N(X) + S\|_\infty \geq \|S\|_\infty$.

**Lemma 2.2.** Let $N \in \mathcal{L}(H)$ be a normal operator and let $H_1 = \text{Vect}_{\lambda \in \mathbb{C}} \ker(N - \lambda)$. If $S \in \{N\}'$ and there exists $X \in \mathcal{L}(H)$ such that $\delta_N(X) + S \in \mathcal{H}(H)$, then $H_1$ reduces $S$ and the restriction $S|_{H_1^\perp} = 0$.

**Proof.** Since $N$ is a normal operator, $H_1$ reduces $N$ and the restriction $N|_{H_1}$ is a diagonal operator, then the Putnam-Fuglede’s theorem guarantees that $S^* \in \{N\}'$. Hence, $H_1$ reduces $S$. Let

$$
N = 
\begin{bmatrix}
N_1 & 0 \\
0 & N_2
\end{bmatrix}, \quad S = 
\begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix}, \quad X = 
\begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix}
$$

(2.5)

on $H = H_1 \oplus H_2$, where $H_2 = H_1^\perp$. The hypothesis $\delta_N(X) + S \in \mathcal{H}(H)$ would imply that $\delta_{N_2}(X_{22}) + S_2 \in \mathcal{H}(H)$. The result of Anderson [1] (applied to the Calkin algebra $\mathcal{L}(H_2) \setminus \mathcal{H}(H_2)$) guarantees that $S_2 \in \mathcal{H}(H)$. Since the normal operator $N_2$ is without eigenvalues and the selfadjoint operator $S_2^*S_2$ is compact and belongs to the commutant of $N_2$, it results that $S_2^*S_2 = 0$ and thus $S_2 = 0$. 

$\square$
THEOREM 2.3. Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator, $S \in \{N\}'$, and $X \in \mathcal{L}(\mathcal{H})$. If $\delta_N(X) + S \in \mathcal{H}(\mathcal{H})$, then $S \in \mathcal{H}(\mathcal{H})$ and

$$||\delta_N(X) + S||_\infty \geq ||S||_\infty.$$  \hspace{1cm} (2.6)

PROOF. Since $\delta_N(X) + S \in \mathcal{H}(\mathcal{H})$, it follows from Lemma 2.2 that

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \quad (2.7)$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$, where $\mathcal{H}_1 = \text{Vect}_{\lambda \in \mathbb{C}} \ker(N - \lambda)$. If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \quad (2.8)$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$, then

$$\delta_N(X) + S = \begin{bmatrix} \delta_{N_1}(X_{11}) + S_1 & * \\ * & * \end{bmatrix}. \quad (2.9)$$

Since $\delta_N(X) + S \in \mathcal{H}(\mathcal{H})$, it results that $\delta_{N_1}(X_{11}) + S_1 \in \mathcal{H}(\mathcal{H})$. As $N$ is a diagonal operator and $S_1 \in \{N_1\}'$, it follows from Lemma 2.1 that $S_1$ is compact and

$$||\delta_{N_1}(X_{11}) + S_1||_\infty \geq ||S_1||_\infty. \quad (2.10)$$

Consequently, $S$ is compact and

$$||\delta_N(X) + S||_\infty \geq ||\delta_{N_1}(X_{11}) + S_1||_\infty \geq ||S_1||_\infty = ||S||_\infty. \quad (2.11)$$

Corollary 2.4. Let $N, M, S \in \mathcal{L}(\mathcal{H})$ such that $N$ and $M$ are normal operators and $NS = SM$. If $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_{N,M}(X) + S \in \mathcal{H}(\mathcal{H})$, then $S \in \mathcal{H}(\mathcal{H})$ and

$$||\delta_{N,M}(X) + S||_\infty \geq ||S||_\infty. \quad (2.12)$$

PROOF. Consider the operators $L$, $T$, and $Y$ defined on $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}$ by

$$L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \quad S = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}, \quad (2.13)$$

then $L$ is normal, $T \in \{L\}'$ and

$$\delta_L(Y) + T = \begin{bmatrix} 0 & \delta_{N,M}(X) + S \\ 0 & 0 \end{bmatrix}. \quad (2.14)$$

Then Theorem 2.3 would imply that $T$ is compact and

$$||\delta_L(Y) + T||_\infty \geq ||T||_\infty. \quad (2.15)$$
consequently, $S$ is compact and
\[ ||\delta_{N,M}(X) + S||_\infty \geq ||S||_\infty. \] (2.16)

3. Generalized derivations. In this section, we generalize the above results to a large class of operators. We show that if the pair $(A,B)$ has the property $(PF)_{\mathcal{K}(\mathcal{H})}$, and $AS = SB$ such that $\delta_{N,M}(X) + S \in \mathcal{K}(\mathcal{H}_n)$, then $S \in \mathcal{K}(\mathcal{H}_n)$ and
\[ ||\delta_{A,B}(X) + S||_\infty \geq ||S||_\infty, \quad \forall X \in \mathcal{L}(\mathcal{H}). \] (3.1)

Before proving this result, we need the following lemma.

**Lemma 3.1.** Let $A,B \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent:
1. the pair $(A,B)$ has the property $(PF)_{\mathcal{K}(\mathcal{H})}$;
2. if $AT = TB$, where $T \in \mathcal{K}(\mathcal{H}_n)$, then $\overline{R(T)}$ reduces $A$, ker$(T)^\perp$ reduces $B$, and $A|_{\overline{R(T)}}$, and $B|_{\text{ker}(T)^\perp}$ are normal operators.

**Proof.** (1)$\Rightarrow$(2). Since $\mathcal{K}(\mathcal{H})$ is a bilateral ideal and $T \in \mathcal{K}(\mathcal{H}_n)$, then $AT \in \mathcal{K}(\mathcal{H})$. Hence, as $AT = TB$ and $(A,B)$ satisfies $(PF)_{\mathcal{K}(\mathcal{H})}$, $A^*T = TB^*$ and $R(T)$, and ker$(T)^\perp$ are reducing subspaces for $A$ and $B$, respectively. Since $A(AT) = (AT)B$ implies $A^*(AT) = (AT)B^*$ by $(PF)_{\mathcal{K}(\mathcal{H})}$, and the identity $A^*T = TB^*$ implies that $A^*AT = AA^*T$, thus we see that $A|_{\overline{R(T)}}$ is normal. Clearly, $(B^*,A^*)$ satisfies $(PF)_{\mathcal{K}(\mathcal{H})}$ and $B^*T^* = T^*A^*$. Therefore, it follows from the above argument that $B^*|_{\overline{R(T)}} = B|_{\text{ker}(T)^\perp}$ is normal.

(2)$\Rightarrow$(1). Let $T \in \mathcal{K}(\mathcal{H})$ such that $AT = TB$. Taking the two decompositions of $\mathcal{H}$, $\mathcal{H}_1 = \mathcal{H} = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $\mathcal{H}_2 = \mathcal{H} = \text{ker}(T)^\perp \oplus \text{ker}T$. Then we can write $A$ and $B$ on $\mathcal{H}_1$ into $\mathcal{H}_2$, respectively,
\[ A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \] (3.2)
where $A_1$ and $B_1$ are normal operators. Also we can write $T$ and $X$ on $\mathcal{H}_2$ into $\mathcal{H}_1$
\[ T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \] (3.3)

It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since $A_1$ and $B_1$ are normal operators, then, by applying the Fuglede-Putnam's theorem, we obtain $A_1^*T_1 = T_1B_1^*$, that is, $A^*T = TB^*$. \qed

**Theorem 3.2.** Let $A,B \in \mathcal{L}(\mathcal{H})$ satisfying $(PF)_{\mathcal{K}(\mathcal{H})}$ and $AS = SB$. If $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_{A,B}(X) + S \in \mathcal{K}(\mathcal{H}_n)$, then $S \in \mathcal{K}(\mathcal{H}_n)$ and
\[ ||\delta_{A,B}(X) + S||_\infty \geq ||S||_\infty. \] (3.4)
Proof. Since the pair \((A, B)\) satisfies the property \((PF)_{\mathcal{H}(\mathcal{H})}\), it follows by Lemma 3.1 that \(R(S)\) reduces \(A\), ker\((S)\) reduces \(B\), and \(A|_{R(S)}\) and \(B|_{\ker(S)}\) are normal operators. Let \(\mathcal{H}_1 = R(S) \oplus R(S)^\perp\) and \(\mathcal{H}_2 = \ker(S)^\perp \oplus \ker S\). Then

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},
\]

\[
S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.
\]

It follows from

\[
AS - SB = \begin{bmatrix} A_1 S_1 - S_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} = 0
\]

(3.5)

that \(A_1 S_1 = S_1 B_1\) and we have

\[
\|S - (AX - XB)\|_\infty = \left\| \begin{bmatrix} S_1 & (A_1 X_1 - X_1 B_1) \\ * & * \end{bmatrix} \right\|_\infty.
\]

(3.7)

Since \(A_1\) and \(B_1\) are two normal operators, then it results from Corollary 2.4 that \(S_1\) is compact and

\[
\|S_1 - (A_1 X_1 - X_1 B_1)\|_\infty \geq \|S_1\|_\infty,
\]

(3.8)

so

\[
\|S - (AX - XB)\|_\infty \geq \|S_1 - (A_1 X_1 - X_1 B_1)\|_\infty \geq \|S_1\|_\infty = \|S\|_\infty.
\]

(3.9)

Corollary 3.3. Let \(A, B \in \mathcal{L}(\mathcal{H})\) satisfying \((PF)_{\mathcal{H}(\mathcal{H})}\) and \(AS = SB\). If \(X \in \mathcal{L}(\mathcal{H})\) such that \(\delta_{A,B}(X) + S \in \mathcal{H}(\mathcal{H})\), then \(S \in \mathcal{H}(\mathcal{H})\) and

\[
\|S + AX - XB\|_\infty \geq \|S\|_\infty
\]

(3.10)

in each of the following cases:

(1) if \(A, B \in \mathcal{L}(\mathcal{H})\) such that \(\|Ax\| \geq \|x\| \geq \|Bx\|\) for all \(x \in \mathcal{H}\);

(2) if \(A\) is invertible and \(B\) such that \(\|A^{-1}\||B\| \leq 1\).

Proof. (1) The result of Tong [5, Lemma 1] guarantees that the above condition implies that for all \(T \in \ker(\delta_{A,B} | \mathcal{H}(\mathcal{H}))\), \(R(T)\) reduces \(A\), ker\((T)\) reduces \(B\), and \(A|_{R(T)}\) and \(B|_{\ker(T)}\) are unitary operators. Hence, it results from Lemma 3.1 that the pair \((A, B)\) has the property \((PF)_{\mathcal{H}(\mathcal{H})}\) and the result holds by Theorem 3.2.

Inequality (3.10) holds in particular if \(A = B\) is isometric; in other words, \(\|Ax\| = \|x\|\) for all \(x \in \mathcal{H}\).

(2) In this case, it suffices to take \(A_1 = \|B\|^{-1} A\) and \(B_1 = \|B\|^{-1} B\), then \(\|A_1 x\| \geq \|x\| \geq \|B_1 x\|\) and the result holds by (1) for all \(x \in \mathcal{H}\).
References


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