MORPHISMS OF MISLIN GENERA INDUCED
BY FINITE NORMAL SUBGROUPS

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We correct an erroneous statement about induced morphisms of Mislin genera and give the correct statement, even under more general hypotheses.

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As in [9], we denote the class of all finitely generated groups with finite commutator subgroups by $\mathcal{X}_0$, and for an $\mathcal{X}_0$-group $H$, we let $\chi(H)$ be the set of isomorphism classes of groups $K$ for which $K \times \mathbb{Z} \cong H \times \mathbb{Z}$. If $H$ is a nilpotent $\mathcal{X}_0$-group, the Mislin genus (i.e., the genus as defined in [4]) of $H$ is denoted by $\mathcal{G}(H)$. By a result of Warfield [6], we know that if $H$ is a nilpotent $\mathcal{X}_0$-group, then $\chi(H) = \mathcal{G}(H)$. Furthermore, for an $\mathcal{X}_0$-group $H$, it is shown in [9] that there is an abelian group structure on $\chi(H)$ which coincides with the Hilton-Mislin group structure [3] on $\mathcal{G}(H)$ if $H$ is nilpotent.

In [8, Section 3], it was shown how to define a function $\eta : \chi(H) \to \chi(H/F)$ if $H$ is an infinite $\mathcal{X}_0$-group and $F$ is a finite normal subgroup of $H$. It was also shown that the function is not always a homomorphism [8, Example 5.4]. This is in conflict with [2, Theorem 1.3]. In fact there is an error in [2, Theorem 1.1] in that the function $\alpha_* : \mathcal{G}(N) \to \mathcal{G}(N/F)$ is not always well defined. The counterexample of [9] suggests a way to show explicitly how things may go wrong. (To merely show that $\alpha_*$ is not always well defined there are simpler examples, but for a simpler example one may find that there is nevertheless some epimorphisms $\mathcal{G}(N) \to \mathcal{G}(N/F)$.) We will show that the results of [2, Section 1] remain valid.

In order to ensure that the relation $\alpha_*$ of [2, Section 1] is a well-defined function, we could follow the option of replacing the domain $\mathcal{G}(N)$ with a different set, which we briefly describe as follows.

Let $\mathcal{X}_0$ be the subclass of $\mathcal{X}_0$ consisting of all infinite nilpotent groups. For an $\mathcal{X}_0$-group $H$ and a suitable finite group $F$, we fix a monomorphism $h : F \to h(F) \triangleleft H$. Now let $K$ be a group in the Mislin genus of $H$, and let $k : F \to K$ be any monomorphism with $k(F) \triangleleft K$ which admits, for every prime $p$, an isomorphism $f : K_p \to H_p$ for which $f \circ k_p = h_p$. We denote the class of all such pairs $(K, k)$ by $\mathcal{X}_0$. If $l : F \to L$ is another such homomorphism, then we say that $l \sim k$ if there is an isomorphism $\phi : L \to K$ for which $\phi \circ l = k$. Then $\sim$ is an equivalence relation. Let $\mathcal{G}(H, h)$ be the set $\mathcal{G}(H, h) = \mathcal{X}_0/\sim$ of all equivalence classes of such endomorphisms. Since $\mathcal{G}(H)$ is finite and since there are only finitely many embeddings of $F$ into $H$, it is easy to prove that $\mathcal{G}(H, h)$ is a finite set. At least then we can follow [2, Theorem 1.1]. The association $(K, k) \mapsto K/k(F)$ determines a function $\alpha_* : \mathcal{G}(H, h) \to \mathcal{G}(H/h(F))$. There is of course the difficulty that
the set \( \mathcal{G}(H, h) \) is not well understood, for example, we do not know whether \( \mathcal{G}(H, h) \) has a suitable group structure. Anyway, we are interested in \( \mathcal{G}(H) \), and we will follow a different option.

We know (see, e.g., [7]) that if \( F \) is a characteristic subgroup of the torsion subgroup \( T_H \) of \( H \), then we do have a homomorphism \( \mathcal{G}(H) \to \mathcal{G}(H/F) \), in fact, an epimorphism. In the calculation that leads up to [2, Theorem 3.1], the subgroup \( \ker \alpha \) of \( N \) that is being factored out is, indeed, a characteristic subgroup of \( T \) (see Proposition 7).

Further we note that \( \tilde{N} \) is of the form \( H \times (\mathbb{Z}_2) \) for some group \( H \), and then by [7, Corollary 4.2] we have an isomorphism \( \mathcal{G}(H) \to \mathcal{G}(\tilde{N}) \). For such a group \( H \) we have (see [1]) that \( \mathcal{G}(H) = (\mathbb{Z}_t)^* / \{1, -1\} \). Thus it follows that [2, Theorem 3.1] is valid. In this paper, we will find a more general condition on the pair \( F \triangleleft H \) in order to have a homomorphism \( \mathcal{G}(H) \to \mathcal{G}(H/F) \), in fact, an epimorphism. Our result in this regard is more general in that we do not require the group \( H \) to be nilpotent.

We recall the following invariant of an \( \mathcal{X}_0 \)-group.

**Definition 1** (see [9]). For an \( \mathcal{X}_0 \)-group \( H \), let \( n_1 \) be the exponent of the torsion subgroup \( T_H \), let \( n_2 \) be the exponent of the group \( \text{Aut}(T_H) \), and let \( n_3 \) be the exponent of the torsion subgroup of the center of \( H \). We define the natural number \( n(H) = n_1 n_2 n_3 \).

Note that if \( H \) is an \( \mathcal{X}_0 \)-group and \( K \) is a group for which \( K \times \mathbb{Z} \cong H \times \mathbb{Z} \), then \( K \) is also an \( \mathcal{X}_0 \)-group and \( T_K \cong T_H \), so that \( n(K) = n(H) \). Also note that for such groups \( H \) and \( K \), if \( \epsilon : H \to K \) is an embedding then the index \( [K : \epsilon(H)] \) is finite.

**Theorem 2.** Let \( H \) be an infinite \( \mathcal{X}_0 \)-group, and let \( n = n(H) \). Let \( F \) be a finite subgroup of \( H \). The following two conditions are equivalent:

1. given any embedding \( \phi : H \to H \) such that \( [H : \phi(H)] \) is relatively prime to \( n \), \( \phi(F) = F \);
2. if \( L \) is any group for which \( L \times \mathbb{Z} \cong H \times \mathbb{Z} \), and \( \beta_1 \) and \( \beta_2 \) are any two embeddings of \( L \) onto subgroups \( K_1 \) and \( K_2 \), respectively, of \( H \), with both \( [H : K_1] \) and \( [H : K_2] \) relatively prime to \( n \), then \( \beta_1^{-1}(F) = \beta_2^{-1}(F) \).

**Proof.** Assume that condition (1) holds and suppose that we are given \( L, \beta_1, \), and \( \beta_2 \) as in (2). Then \( F \) is contained in both \( K_1 \) and \( K_2 \). In order to prove (2), it suffices to show that, given any isomorphism \( \beta : K_1 \to K_2 \), \( \beta(F) = F \). By [9, Theorem 4.2] it follows that there is an embedding \( \gamma : H \to K_1 \) such that \( [K_1 : \gamma(H)] \) is relatively prime to \( n \) (note that \( n(H) = n(K_1) \)). Let \( \epsilon : K_1 \to H \) and \( \delta : K_2 \to H \) be the inclusions. Then we have embeddings \( \epsilon \circ \gamma \) and \( \delta \circ \beta \circ \gamma \) of \( H \) into \( H \). By (1), it follows that \( \epsilon \circ \gamma(F) = F \) and \( \delta \circ \beta \circ \gamma(F) = F \). Moreover, \( \epsilon(F) = F \) and \( \delta(F) = F \), and consequently we have \( \beta(F) = F \). So we have proved that (1) implies (2).

The converse implication is clear.

**Remark 3.** Notice that for any infinite \( \mathcal{X}_0 \)-group \( H \) and any group \( L \) for which \( L \times \mathbb{Z} \cong H \times \mathbb{Z} \), \( L \) is an \( \mathcal{X}_0 \)-group and \( n(L) = n(H) \). It is then not hard to see that conditions (1) and (2) of Theorem 2 are equivalent to the following condition:

3. if \( \beta_1 \) and \( \beta_2 \) are any two embeddings of \( H \) onto subgroups \( K_1 \) and \( K_2 \), respectively, of \( L \), with \( [L : K_1] \) and \( [L : K_2] \) relatively prime to \( n \), then \( \beta_1(F) = \beta_2(F) \).
We are now able to state and prove a significant result on induced morphisms.

**Theorem 4.** Let $H$ be an $\mathfrak{X}_0$-group, and let $n = n(H)$. Let $F$ be a finite subgroup of $H$ with the property that, given any embedding $\phi : H \to H$ such that $[H : \phi(H)]$ is relatively prime to $n$, $\phi(F) = F$. Then, for subgroups $K$ of $H$ with $[H : K]$ relatively prime to $n$, the association $K \mapsto K/F$ defines an epimorphism $\eta : \chi(H) \to \chi(H/F)$.

**Proof.** We first note that, by implication, $F$ must be a normal subgroup of $H$. By the equivalence of (1) and (2) in Theorem 2, it follows that $\eta$ is well defined. The proof is completed in a way similar to the proof of [7, Theorem 2.1] using [9, Proposition 6.1].

For an $\mathfrak{X}_0$-group $H$, $T_H$ has finite characteristic subgroups $[T_H, T_H]$ and $Z(T_H)$ to which [7, Theorem 2.1] applies. We point out some other subgroups to which the more general Theorem 4 is applicable.

**Theorem 5.** Let $H$ be an infinite $\mathfrak{X}_0$-group. Let $F = [H, H] \cap T_H$. Then $H$, together with $F$, satisfies condition (1) of Theorem 2.

**Proof.** Let $\phi : H \to H$ be any embedding such that $[H : \phi(H)]$ is relatively prime to $n$. Then $\phi[H, H] = [\phi H, \phi H] < [H, H]$. Also $\phi(T_H) < T_H$. Thus $\phi(F) < F$. Since $F$ is finite, it follows that $\phi(F) = F$.

**Theorem 6.** Let $H$ be an infinite $\mathfrak{X}_0$-group. Let $F = ZH \cap T_H$. Then $H$ together with $F$ satisfies condition (1) of Theorem 2.

**Proof.** Let $\phi : H \to H$ be any embedding such that $[H : \phi(H)]$ is relatively prime to $n$. Then $\phi$ can be extended to an isomorphism $\psi : H \times \mathbb{Z}^k \to H \times \mathbb{Z}^k$ for some $k \in \mathbb{N}$ (see the proof of [9, Theorem 4.1]). Now $Z(H \times \mathbb{Z}^k) = (ZH) \times \mathbb{Z}^k$. Since the isomorphism $\psi$ preserves centers and preserves torsion, it follows that $\psi(F) = F$. Since the induced homomorphism $\phi$ maps $T_H$ isomorphically onto $T_H$, it follows that $\phi(F) = F$.

The following result offers an alternative approach to [2, Theorem 3.1], or to a generalization of it.

**Proposition 7.** Let $n \in \mathbb{N}$, and let

$$T = \langle x, y, z \mid x^2 = y^2 = z^{2n} = 1, [x, y] = z^n, [x, z] = 1 = [y, z] \rangle.$$  \hspace{1cm} (1)

Then the subgroup $F = \langle x, y, z^n \rangle$ of $T$ is a characteristic subgroup of $T$.

**Proof.** We note that $F$ is generated by elements of order 2 and every element of order 2 in $T$ is contained in $F$. Therefore $F$ is a characteristic subgroup of $T$.

**Proposition 8.** Let $n, u \in \mathbb{N}$ be such that $u$ is relatively prime to $2n$. Let $t$ be the multiplicative order of $u \mod 2n$, and let $\tilde{t}$ be the multiplicative order of $u \mod n$. Let $T$ and $F$ be the groups of Proposition 7, and let $\zeta$ be the action of $\mathbb{Z}$ on $T$ defined (for $a \in \mathbb{Z}$) by

$$\zeta(a, z) \mapsto z^{(u^a)}, \quad (a, x) \mapsto x, \quad (a, y) \mapsto y.$$  \hspace{1cm} (2)
Then, for the group $H = T \rtimes \mathbb{Z}$, $F \triangleleft H$ and we have an epimorphism $\chi(H) \to \chi(H/F) = (\mathbb{Z}_i)^* / \{1, -1\}$.

In particular, if $\tilde{t} = t$, then $\chi(H) \simeq \chi(H/F)$.

**Proof.** Our conditions ensure that indeed $\zeta$ is an action. By Proposition 7, $F$ is a characteristic subgroup of $T$, and thus by Theorem 4, there is an epimorphism $\chi(H) \to \chi(H/F)$. The group $H/F$ is isomorphic to the group

$$\langle a, b \mid a^n = 1, \ b a b^{-1} = a^{-1} \rangle$$

and therefore by [5, Theorem 3.8] we have $\chi(H/F) = (\mathbb{Z}_i)^*/\{1, -1\}$.

By [8, Theorem 2.6] there is an epimorphism

$$(\mathbb{Z}_i)^*/\{1, -1\} \to \chi(H),$$

and so, if $\tilde{t} = t$, then $\chi(H) \simeq \chi(H/F)$.

**References**


