ON INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

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We obtain necessary and (different) sufficient conditions for a series summable $|\tilde{N}, p_n|_k$, $1 < k \leq s < \infty$, to imply that the series is summable $|T|_s$, where $(\tilde{N}, p_n)$ is a weighted mean matrix and $T$ is a lower triangular matrix. As corollaries of this result, we obtain several inclusion theorems.

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Let $\sum a_n$ be a given series with partial sums $s_n$, $(C, \alpha)$ the Césaro matrix of order $\alpha$. If $\sigma_n^\alpha$ denotes the $n$th term of the $(C, \alpha)$-transform of $\{s_n\}$ then, from Flett [4], $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ if

$$\sum_{n=1}^\infty n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty.$$  \hspace{1cm} (1)

For any sequence $\{u_n\}$, the forward difference operator $\Delta$ is defined by $\Delta u_n = u_n - u_{n-1}$.

An appropriate extension of (1) to arbitrary lower triangular matrices $T$ is

$$\sum_{n=1}^\infty n^{k-1} |\Delta t_{n-1}|^k < \infty,$$  \hspace{1cm} (2)

where

$$t_n := \sum_{k=0}^n t_{nk}s_k.$$  \hspace{1cm} (3)

Such an extension is used, for example, in Bor [2]. But Sarigöl, Sulaiman, and Bor and Thorpe [3] make the following extension of (1).

They define a series $\sum a_n$ to be summable $|\tilde{N}, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^\infty \left(\frac{p_n}{p_{n-1}}\right)^{k-1} |\Delta Z_{n-1}|^k < \infty,$$  \hspace{1cm} (4)

where $Z_n$ denotes the $n$th term of the weighted mean transform of $\{s_n\}$; that is,

$$Z_n = \frac{1}{p_n} \sum_{k=0}^n p_k s_k.$$  \hspace{1cm} (5)

Apparently they have interpreted the $n$ in (1) to represent the reciprocal of the $n$th diagonal term of the matrix $(\tilde{N}, p_n)$. (See, e.g., Sarigöl [6], where this is explicitly the case.)
Unfortunately, this interpretation cannot be correct. For if it were, then, since the $n$th diagonal entry of $(C, \alpha)$ is $O(n^{-\alpha})$, (1) would take the form
\[
\sum_{n=1}^{\infty} (n^\alpha)^{(k-1)} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right| \sim k < \infty. \tag{6}
\]

However, Flett stays with (1). Thus (2) is an appropriate extension of (1) to lower triangular matrices.

Given any lower triangular matrix $T$, we can associate the matrices $\overline{T}$ and $\hat{T}$ with entries defined by
\[
\overline{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad \hat{t}_{nk} = \overline{t}_{nk} - \overline{t}_{n,k}, \tag{7}
\]
respectively.

Thus, from (3),
\[
t_n = \sum_{k=0}^{n} t_{nk}s_k = \sum_{k=0}^{n} \overline{t}_{nk} \sum_{i=0}^{k} a_i = \sum_{i=0}^{n} a_i \sum_{k=0}^{n} \overline{t}_{nk}a_i, \tag{8}
\]
\[
Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \hat{t}_{ni}a_i - \sum_{i=0}^{n-1} \hat{t}_{n-1,i}a_i = \sum_{i=0}^{n} \hat{t}_{ni}a_i, \quad \text{since } \hat{t}_{n-1,n} = 0.
\]

For a weighted mean matrix $A = (\tilde{N}, p_n)$,
\[
\bar{a}_{nk} = \sum_{i=k}^{n} \frac{p_k}{p_n} = \frac{1}{p_n} (p_n - p_{k-1}) = 1 - \frac{p_{k-1}}{p_n}. \tag{9}
\]

Thus
\[
\hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k} = 1 - \frac{p_{k-1}}{p_n} - 1 + \frac{p_{k-1}}{p_{n-1}} = \frac{p_n p_{k-1}}{p_{n-1} p_{n-1}}, \tag{10}
\]
so that, from (5),
\[
X_n := \Delta Z_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{k=0}^{n-1} p_{k-1}a_k = \frac{p_n}{p_n p_{n-1}} \sum_{\nu=1}^{n-1} p_{\nu-1}a_{\nu}, \tag{11}
\]
since $P_{-1} = 0$.

We will always assume that $\{p_n\}$ is a positive sequence with $P_n \to \infty$. Also, $\Delta \nu \hat{t}_{n\nu} := \hat{t}_{n\nu} - \hat{t}_{n\nu+1}$.

**Theorem 1.** Let $1 < k \leq s < \infty$, $\{p_n\}$ satisfying
\[
\sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{p_n}{p_n p_{n-1}} \right)^{s} = O \left( \frac{1}{p_n^{s/k}} \right). \tag{12}
\]

Let $T$ be a lower triangular matrix. Then, the necessary conditions for $\sum a_n$ summable $|\tilde{N}, p_n|_k$ to imply $\sum a_n$ is summable $|T|_s$ are

(i) $P_\nu |t_{\nu\nu}|/p_\nu = O (\nu^{1/s-1/k})$;
(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \nu \hat{t}_{n\nu}|^s = O (\nu^{s/k} (p_\nu/p_\nu)^s)$;
(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n\nu+1}|^s = O (1)$.
**Proof.** We are given that
\[
\sum_{n=1}^{\infty} n^{s-1} |Y_n|^s < \infty,
\] (13)
whenever
\[
\sum_{n=1}^{\infty} n^{k-1} |X_n|^k < \infty.
\] (14)

Now, the space of sequences \(\{a_n\}\) satisfying (14) is a Banach space if normed by
\[
\|X\| = \left( |X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right)^{1/k}.
\] (15)

We also consider the space of those sequences \(\{Y_n\}\) that satisfy (13). This is also a BK-space with respect to the norm
\[
\|Y\| = \left( |Y_0|^s + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right)^{1/s}.
\] (16)

Observe that (8) transforms the space of sequences satisfying (14) into the space of sequences satisfying (13). Applying the Banach-Steinhaus theorem, there exists a constant \(K > 0\) such that
\[
\|Y\| \leq K \|X\|.  
\] (17)

Applying (11) and (8) to \(a_v = e_v - e_{v+1}\), where \(e_v\) is the \(v\)th coordinate vector, we have, respectively,
\[
X_n = \begin{cases} 
0, & \text{if } n < v, \\
p_v, & \text{if } n = v, \\
-p_v p_n & \text{if } n > v, \\
p_v p_n / p_{n-1}, & \text{if } n = v, \\
-1 & \text{if } n > v,
\end{cases}
\] (18)
\[
Y_n = \begin{cases} 
0, & \text{if } n < v, \\
\hat{t}_{nv}, & \text{if } n = v, \\
\Delta_v \hat{t}_{nv} & \text{if } n > v.
\end{cases}
\]

By (15) and (16), it follows that
\[
\|X\| = \left( \nu^{k-1} \left( \frac{p_v}{P_v} \right)^k + \sum_{n=1}^{\nu} n^{k-1} \left( \frac{p_v p_n}{P_n P_{n-1}} \right)^k \right)^{1/k},
\] (19)
\[
\|Y\| = \left( \nu^{s-1} |t_{vv}|^s + \sum_{n=1}^{\nu} n^{s-1} |\Delta_v \hat{t}_{nv}|^s \right)^{1/s},
\] (20)
recalling that \(\hat{t}_{vv} = \bar{t}_{vv} = t_{vv}\).
Using (19) and (20) in (17), along with (12), it follows that

\[
\nu^{s-1} |t_{vv}|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{nv}|^s \leq K^s \left( \nu^{k-1} \left( \frac{p_{\nu}}{P_{\nu}} \right)^k + \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{P_{\nu}}{P_n P_{n-1}} \right)^k \right)^{s/k}
\]

\[
\leq K^s \left( \nu^{k-1} \left( \frac{p_{\nu}}{P_{\nu}} \right)^k + O(1) \left( \frac{p_{\nu}}{P_{\nu}} \right)^k \right)^{s/k}
\]

\[
= O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^k \nu^{k-1} \right)^{s/k}.
\]

The above inequality will be true if and only if each term on the left-hand side is \(O((p_{\nu}/P_{\nu})^k \nu^{k-1})^{s/k}\).

Taking the first term, we have,

\[
\nu^{s-1} |t_{vv}|^s = O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^k \nu^{k-1} \right)^{s/k},
\]

\[
|t_{vv}|^s = O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^s \nu^{1-s/k} \right),
\]

\[
|t_{vv}| = O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^s \nu^{1-s/k} \right)^{1/s}
\]

\[
= O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^s \nu^{1/s-1/k} \right),
\]

which verifies that (i) is necessary.

Using the second term, we have

\[
\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta \hat{t}_{nv}|^s = O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^k \nu^{k-1} \right)^{s/k} = O \left( \left( \frac{p_{\nu}}{P_{\nu}} \right)^s \nu^{s-1/k} \right),
\]

which is condition (ii).

If we now apply (11) and (8) to \(a_{\nu} = e^{\nu+1}\), we have, respectively,

\[
X_n = \begin{cases} 
0, & \text{if } n \leq \nu, \\
\frac{P_{\nu} \nu_n}{P_n P_{n-1}}, & \text{if } n > \nu,
\end{cases}
\]

\[
Y_n = \begin{cases} 
0, & \text{if } n \leq \nu, \\
\hat{t}_{n,\nu+1}, & \text{if } n > \nu.
\end{cases}
\]
ON INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

The corresponding norms are

\[
\|X\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{P_{\nu}P_n}{P_nP_{\nu-1}} \right)^k \right\}^{1/k},
\]

\[
\|Y\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \right\}^{1/s},
\]

(25)

Applying (17) and (12),

\[
\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s \leq K^s \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{P_{\nu}P_n}{P_nP_{\nu-1}} \right)^k \right\}^{s/k},
\]

(26)

which is condition (iii).

COROLLARY 2. Let \( T \) be a lower triangular matrix, \( \{p_n\} \) satisfying (12). Then the necessary conditions for \( \sum a_n \) summable \( |\tilde{N},p_n|_k \) to imply \( \sum a_n \) summable \( |T|_k \) are

(i) \( P_{\nu}|t_{\nu \nu}|/p_{\nu} = O(1) \);
(ii) \( \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_{\nu} \hat{t}_{\nu \nu}|^k = O(v^{k-1}(p_{\nu}/P_{\nu})^k) \);
(iii) \( \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu+1}|^k = O(1) \).

To prove Corollary 2, simply set \( s = k \) in Theorem 1.

A lower triangular matrix \( T \) is called a triangle if each \( t_{nn} \neq 0 \).

THEOREM 3. Let \( 1 < k \leq s < \infty \). Let \( T \) be a triangle with bounded entries such that \( T \) and \( \{p_n\} \) satisfy the following:

(i) \( t_{\nu \nu} = O((p_{\nu}/P_{\nu})^{1/s-1/k}) \);
(ii) \( (n|X_n|)^{s-k} = O(1) \);
(iii) \( \sum_{n=1}^{\nu-1} |\Delta_{\nu} \hat{t}_{\nu \nu}| = O(|t_{nn}|) \);
(iv) \( \sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1} |\Delta_{\nu} \hat{t}_{\nu \nu}| = O(v^{s-1}|t_{\nu \nu}|^s) \);
(v) \( \sum_{n=\nu+1}^{\infty} |t_{\nu \nu}| |\hat{t}_{n,\nu+1}| = O(|t_{nn}|) \);
(vi) \( \sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1} |\hat{t}_{n,\nu+1}| = O(v|t_{\nu \nu}|)^{s-1} \).

Then \( \sum a_n \) is \( \tilde{N},p_n|_k \).

PROOF. Solving (11) for \( \{a_n\} \) and substituting into (8) give

\[
Y_n = \sum_{\nu=1}^{n} \hat{t}_{\nu \nu} \left( \frac{X_{\nu}P_{\nu}}{p_{\nu}} - \frac{X_{\nu-1}P_{\nu-2}}{p_{\nu-1}} \right)
\]

\[
= \sum_{\nu=1}^{n} \hat{t}_{\nu \nu} \frac{X_{\nu}P_{\nu}}{p_{\nu}} - \sum_{\nu=1}^{n} \hat{t}_{\nu \nu} \frac{X_{\nu-1}P_{\nu-2}}{p_{\nu-1}}
\]

\[
= \sum_{\nu=1}^{n} \hat{t}_{\nu \nu} \frac{X_{\nu}P_{\nu}}{p_{\nu}} - \sum_{\nu=1}^{n} \hat{t}_{\nu \nu} \frac{X_{\nu-1}P_{\nu-2}}{p_{\nu-1}}
\]

\[
= \hat{t}_{nn}X_nP_n + \sum_{\nu=1}^{n-1} (\hat{t}_{\nu \nu}P_{\nu} - \hat{t}_{\nu \nu+1}P_{\nu-1}) \frac{X_{\nu}}{p_{\nu}}
\]
\[
\begin{align*}
= t_{nn}P_nX_n/p_n + \sum_{v=1}^{n-1} \left[ P_v(t_{nv} - \hat{t}_{n,v+1}) + \hat{t}_{n,v+1}(P_v - P_{v-1}) \right]X_v/p_v \\
= P_n t_{nn}X_n/p_n + \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1} \right)X_v \\
= T_{n1} + T_{n2} + T_{n3}.
\end{align*}
\]

(27)

From Minkowski's inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} n^{s-1} \mid T_{ni} \mid^s < \infty, \quad i = 1, 2, 3.
\]

(28)

Using condition (i) of Theorem 3,

\[
J_1 := \sum_{n=1}^{\infty} n^{s-1} \mid T_{n1} \mid^s = \sum_{n=1}^{\infty} n^{s-1} \mid \frac{t_{nn}P_n}{p_n}X_n \mid^s
\]

\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} (n^{1/s - 1/k})^s \mid X_n \mid^s
\]

\[
= O(1) \sum_{n=1}^{\infty} n^{k-1} \mid X_n \mid^k \left( n^{s-s/k-k+1} \mid X_n \mid^{s-k} \right).
\]

(29)

But

\[
n^{s-s/k-k+1} \mid X_n \mid^{s-k} = (n^{1-k} \mid X_n \mid)^{s-k} = O \left( (n \mid X_n \mid)^{s-k} \right) = O(1),
\]

(30)

from (ii) of Theorem 3.

Since \( \sum a_n \) is summable, \( |\tilde{N},p_n|, J_1 = O(1) \).

Using Hölder’s inequality and conditions (i), (ii), (iii), and (iv) of Theorem 3.

\[
J_2 := \sum_{n=1}^{\infty} n^{s-1} \mid T_{n2} \mid^s = \sum_{n=1}^{\infty} n^{s-1} \mid \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right) (\Delta_v \hat{t}_{nv})X_v \mid^s
\]

\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{v=1}^{n-1} v^{1/s - 1/k} \mid t_{nv} \mid^{-1} \mid \Delta_v \hat{t}_{nv} \mid \mid X_v \mid \right)^s
\]

\[
= O(1) \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{v=1}^{n-1} v^{1-s/k} \mid t_{nv} \mid^{-s} \mid \Delta_v \hat{t}_{nv} \mid \mid X_v \mid^s \right) \times \left( \sum_{v=1}^{n-1} \mid \Delta_v \hat{t}_{nv} \mid \right)^{s-1}
\]

\[
= O(1) \sum_{n=1}^{\infty} (n \mid t_{nn} \mid)^{s-1} \sum_{v=1}^{n-1} v^{1-s/k} \mid t_{nv} \mid^{-s} \mid \Delta_v \hat{t}_{nv} \mid \mid X_v \mid^s
\]

\[
= O(1) \sum_{v=1}^{\infty} v^{1-s/k} \mid t_{vv} \mid^{-s} \mid X_v \mid^s \sum_{n=v+1}^{\infty} (n \mid t_{nn} \mid)^{s-1} \mid \Delta_v \hat{t}_{nv} \mid
\]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{s-1/k} |t_{\nu \nu}|^{-s} |X_\nu|^s |t_{\nu \nu}|^s \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{s-1/k} |X_\nu|^s \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_\nu|^k (\nu^{s-1/k} |X_\nu|^{s-k}) \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_\nu|^k = O(1). \]

(31)

By Hölder's inequality and conditions (v), (vi), and (iii) of Theorem 3, we have

\[ J_3 := \sum_{n=1}^{\infty} n^{s-1} |T_{n3}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=1}^{n-1} \hat{t}_{n,\nu+1} X_\nu \right|^s \]
\[ \leq \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{\nu=1}^{n-1} \left| \hat{t}_{n,\nu+1} \right| |X_\nu| \right)^s \]
\[ \leq \sum_{n=1}^{\infty} n^{s-1} \left( \sum_{\nu=1}^{n-1} |t_{\nu \nu}|^{1-s} \left| \hat{\nu}_{n,\nu+1} \right| |X_\nu|^s \right) \]
\[ \times \left( \sum_{\nu=1}^{n-1} |t_{\nu \nu}| \left| \hat{\nu}_{n,\nu+1} \right| \right)^{s-1} \]
\[ = O(1) \sum_{n=1}^{\infty} (n |t_{nn}|)^{s-1} \sum_{\nu=1}^{n-1} |t_{\nu \nu}|^{1-s} \left| \hat{\nu}_{n,\nu+1} \right| |X_\nu|^s \]
\[ = O(1) \sum_{\nu=1}^{\infty} |t_{\nu \nu}|^{1-s} |X_\nu|^s \sum_{n=\nu+1}^{\infty} (n |t_{nn}|)^{s-1} \left| \hat{\nu}_{n,\nu+1} \right| \]
\[ = O(1) \sum_{\nu=1}^{\infty} |t_{\nu \nu}|^{1-s} |X_\nu|^s (\nu |t_{\nu \nu}|)^{s-1} \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{s-1} |X_\nu|^s \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_\nu|^k (\nu |X_\nu|)^{s-k} \]
\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_\nu|^k = O(1). \]

(32)

**Corollary 4** (see [5]). Let \( T \) be a nonnegative lower triangular matrix, \( \{p_n\} \) a positive sequence satisfying

(i) \( t_{ni} \geq t_{n+1,i}, n \geq i, i = 0, 1, 2, \ldots; \)

(ii) \( p_n t_{nn} = O(p_n); \)

(iii) \( \hat{t}_{n0} = \hat{t}_{n-1,0}, n = 1, 2, \ldots; \)
(iv) $\sum_{i=1}^{n-1} |t_{ii}| \Delta i_{n,i+1} = O(t_{nn})$;
(v) $\sum_{n=1}^{\infty} (n t_{nn})^{k-1} |\Delta i_{n}| = O((i_{nn})^{k-1})$;
(vi) $\sum_{n=1}^{\infty} (n t_{nn})^{k-1} i_{n,i+1} = O((i_{nn})^{k-1})$.

Then $\sum a_n$ summable $|N, p_n| = k$ implies $\sum a_n$ summable $|T| = k$, $k \geq 1$.

**Proof.** Since $s = k$ and $T$ is nonnegative, condition (ii) of Theorem 3 is automatically satisfied, and conditions (ii), (iv), (v), and (vi) of Corollary 4 are equivalent to conditions (i), (v), (iv), and (vi) of Theorem 3, respectively

\[ \Delta y i_{n,y} = i_{n,y} - i_{n,y+1} = i_{n,y} - i_{n,y+1} + i_{n,y+2} = n_{n,y} - n_{n,y+1}. \]

Therefore, using conditions (i) and (iii) of Corollary 4,

\[ \sum_{y=1}^{n-1} |\Delta y i_{n,y}| = \sum_{y=1}^{n-1} (n_{n,y-1} - n_{n,y}) = 1 - t_{n-1,0} - 1 + t_{n,n} + t_{n,0} \leq t_{nn}, \]

and condition (iii) of Theorem 3 is satisfied.

**Remark 5.** For $1 < k \leq s < \infty$, necessary and sufficient conditions for a triangle $A : \ell^k \rightarrow \ell^s$ are known only for factorable matrices (see Bennett [1]), which include weighted mean matrices. Therefore, we should not expect to obtain a set of necessary and sufficient conditions when an arbitrary triangle is involved.

However, necessary and sufficient conditions for a matrix $A : \ell \rightarrow \ell^s$, $1 \leq s < \infty$ are known. The following result comes from Theorem 2.1 of Rhoades and Savaş [5] by setting each $\lambda_n = 1$.

**Theorem 6.** Let $T$ be a lower triangular matrix. Then $\sum a_n$ summable $|N, p_n| = k$ implies $\sum a_n$ summable $|T| = s$, $s \geq 1$ if and only if

(i) $P|t_{yy}|/p_y = O(y^{1/s-1})$,
(ii) $\sum_{n=1}^{\infty} n^{s-1} |\Delta y i_{n,y}| = O((n_y/P_y)^s)$,
(iii) $\sum_{n=1}^{\infty} n^{s-1} i_{n,y+1} = O(1)$.

**Remark 7.** In [5], it is assumed that $T$ has nonnegative entries and row sums one, but these restrictions are not used in the proofs.

Finally, we state necessary and sufficient conditions when $k = s = 1$.

**Theorem 8.** The series $\sum a_n$ summable $|N, p_n|$ implies $\sum a_n$ summable $T$ if and only if

(i) $P|t_{yy}|/p_y = O(1)$;
(ii) $\sum_{n=1}^{\infty} |\Delta y i_{n,y}| = O(p_y/P_y)$;
(iii) $\sum_{n=1}^{\infty} i_{n,y+1} = O(1)$.

**Proof.** Note that, with $k = 1$, (12) is automatically satisfied. Therefore, the necessity of the conditions follows from Theorem 1.

To prove the conditions sufficient, use [5, Corollary 4.1] by setting each $\lambda_n = 1$.

**Corollary 9.** $\sum a_n$ summable $|C, 1|$ implies $\sum a_n$ summable $|N, q_n|$ if and only if

(i) $n q_n/Q_n = O(1)$. 

\textbf{Proof.} With each }p_n = 1, T = (\bar{N}, q_n), \text{ condition (i) of Theorem 8 reduces to condition (i) of Corollary 9.

Using (33),

\begin{equation}
\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{t}_{n,\nu}| = \sum_{n=\nu+1}^{\infty} |t_{n,\nu} - t_{n-1,\nu}| = \sum_{n=\nu+1}^{\infty} \left| \frac{p_\nu}{p_n} - \frac{p_\nu}{p_{n-1}} \right| = p_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{p_n p_{n-1}} = \frac{p_\nu}{P_\nu},
\end{equation}

(35)

and condition (ii) of Theorem 8 is satisfied. Since \((\bar{N}, p_n)\) has row sums one,

\begin{equation}
\hat{t}_{n,\nu+1} = \hat{t}_{n,\nu+1} - \hat{t}_{n-1,\nu+1} = \sum_{i=\nu+1}^{n} t_{ni} - \sum_{i=\nu+1}^{n} t_{n-1,i} = 1 - \sum_{i=0}^{\nu} t_{ni} - 1 + \sum_{i=0}^{\nu} t_{n-1,i},
\end{equation}

(36)

Therefore

\begin{equation}
\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = P_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{p_n p_{n-1}} = 1,
\end{equation}

(37)

and condition (iii) of Theorem 8 is satisfied. \(\square\)

\textbf{Corollary 10.} The series }\sum a_n \text{ summable } |\bar{N}, p_n|_k \text{ implies } \sum a_n \text{ summable } |C, 1|_k \text{ if and only if}

(i) }\frac{p_n}{(np_n)} = O(1).

\textbf{Proof.} Using }T = (C, 1) \text{ in Theorem 8, condition (i) of Theorem 8 reduces to condition (i) of Corollary 10.

From (33) and (i) of Corollary 10,

\begin{equation}
\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{t}_{n,\nu}| = \sum_{n=\nu+1}^{\infty} |t_{n-1,\nu} - t_{n,\nu}| = \sum_{n=\nu+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{\nu+1} = \frac{P_\nu}{\nu P_\nu} \left( \frac{\nu}{\nu+1} \right) \left( \frac{P_\nu}{P_\nu} \right) = O \left( \frac{p_\nu}{P_\nu} \right),
\end{equation}

(38)

and condition (ii) of Theorem 8 is satisfied.
Using (36),

$$\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = \sum_{n=\nu+1}^{\infty} \left| \sum_{l=0}^{\nu} (t_{n-1,l} - t_{n,l}) \right|$$

$$= \sum_{n=\nu+1}^{\infty} \left| \sum_{l=0}^{\nu} \left( \frac{1}{n} - \frac{1}{n+1} \right) \right|$$

$$= \sum_{n=\nu+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) (\nu+1) = (\nu+1) \left( \frac{1}{\nu+1} \right) = 1,$$

and condition (iii) of Theorem 8 is satisfied.

Combining Corollaries 9 and 10, we have the following corollary.

**Corollary 11.** $|\bar{N}, p_n|$ and $|C,1|$ are equivalent if and only if

(i) $np_n/P_n = O(1)$;

(ii) $P_n/(np_n) = O(1)$.

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**References**


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