A NOTE ON A CLASS OF BANACH ALGEBRA-VALUED POLYNOMIALS

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Let $F$ be a Banach algebra. We give a necessary and sufficient condition for $F$ to be finite dimensional, in terms of finite type $n$-homogeneous $F$-valued polynomials.

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1. Introduction and results. Let $E$ and $F$ be complex Banach spaces. We denote by $L(nE,F)$ the Banach space of all continuous $n$-linear mappings $A$ from $E^n$ into $F$ endowed with the norm $\|A\| = \sup\{\|A(x_1,\ldots,x_n)\| : \|x_j\| \leq 1, j = 1,\ldots,n\}$. A mapping $P$ from $E$ into $F$ is called a continuous $n$-homogeneous polynomial if $P(x) = A(x,\ldots,x)$ (for all $x \in E$) for some $A \in L(nE,F)$. We denote by $P(nE,F)$ the Banach space of all continuous $n$-homogeneous polynomials $P$ from $E$ into $F$ endowed with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$. Also a mapping $P$ from $E$ into $F$ is called a finite type $n$-homogeneous polynomial if $P(x) = f_1(x)b_1 + \cdots + f_k(x)b_k$ (for all $x \in E$), where $f_1,\ldots,f_k \in E^*$ and $b_1,\ldots,b_k \in F$. We denote by $Pf(nE,F)$ the space of all finite type $n$-homogeneous polynomials $P$ from $E$ into $F$. Then we have $Pf(nE,F) \subseteq P(nE,F)$. Indeed, let $P \in Pf(nE,F)$. Then we write $P(x) = f_1(x)b_1 + \cdots + f_k(x)b_k$ (for all $x \in E$), where $f_1,\ldots,f_k \in E^*$ and $b_1,\ldots,b_k \in F$. Let

$$A(x_1,\ldots,x_n) = \sum_{i=1}^{k} f_i(x_1)\cdots f_i(x_n)b_i, \quad (x_1,\ldots,x_n) \in E^n. \quad (1.1)$$

Then $A$ is a continuous $n$-linear mapping from $E^n$ into $F$ and $P(x) = A(x,\ldots,x)$ (for all $x \in E$). Hence $P \in P(nE,F)$. We are now interested in the case that $F$ is a Banach algebra. Let

$$P_{f}(nE,F) = \{\varphi_1^n + \cdots + \varphi_k^n : \varphi_j \in B(E,F) (j = 1,\ldots,k), k \in \mathbb{N}\}, \quad (1.2)$$

where $\varphi_j^n(x) = (\varphi_j(x))^n (x \in E)$. Then we have $P_{f}(nE,C) = P_{f}(nE,F)$ and $P_{f}(nC,F) \subseteq P_{f}(nE,F)$ (see [1, Section 1]). Also, we have $P_{f}(nE,F) \subseteq P(nE,F)$. Indeed, let $P \in P_{f}(nE,F)$. Then we can write $P = \varphi_1^n + \cdots + \varphi_k^n$ for some $\varphi_1,\ldots,\varphi_k \in B(E,F)$. Set $A(x_1,\ldots,x_n) = \sum_{i=1}^{k} \varphi_i(x_1)\cdots \varphi_i(x_n)$ (for all $x_1,\ldots,x_n \in E^n$). Then $A$ is a continuous $n$-linear mapping from $E^n$ into $F$ and $P(x) = A(x,\ldots,x)$ (for all $x \in E$). Hence $P \in P(nE,F)$.

Now, for each $n \in \mathbb{N}$, we say that an algebra $F$ has the $r_n$-property if, given any $b \in F$, we can find elements $a_1,\ldots,a_p \in F$ such that $b = \sum_{i=1}^{p} a_i^n$. We also say that an algebra $F$ has the $r$-property if $F$ has the $r_n$-property for each $n \in \mathbb{N}$. 
**Proposition 1.1** (see [1]). (1) Every unital complex algebra has the $r$-property.

(2) Let $E$ be a Banach space and $F$ be a Banach algebra. Then $P_f(nE,F) \subseteq P_f(nE,F)$ if and only if $F$ has the $r_n$-property.

In [1], it is remarked that, given an arbitrary Banach space $(F, +, \| \cdot \|)$, we can always define a product $\circ$ and a norm $\| \cdot \|_*$ on $F$ in order that $(F, +, \circ, \| \cdot \|_*)$ is a unital Banach algebra and $\| \cdot \|_*$ is equivalent to $\| \cdot \|$. By Proposition 1.1 and the above remark, Lourenço-Moraes proved the following proposition.

**Proposition 1.2** (see [1]). Let $E$ be a Banach space. The following are equivalent:

(a) $E$ is a finite-dimensional space;

(b) $P_f(nE,F) = P_f(nE,F)$ for every $n \in \mathbb{N}$ and for every Banach algebra $F$ with the $r_n$-property;

(c) $P_f(nE,F) = P_f(nE,F)$ for every $n \in \mathbb{N}$ and for every unital Banach algebra $F$.

**Remark 1.3.** By the proof of Proposition 1.2 (see [1]), we see that each of the following two statements are also equivalent to one of, hence all of, (a), (b), and (c) in Proposition 1.2:

(b') $P_f(1E,F) = P_f(1E,F)$ for every unital Banach algebra $F$;

(d) $P_f(nE,F) = P_f(nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space $F$.

In this note we show the following result, which is opposite to Proposition 1.2.

**Proposition 1.4.** Let $F$ be a Banach algebra. Then the following are equivalent:

(a) $F$ is a finite-dimensional space;

(b) $P_f(nE,F) \subseteq P_f(nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space $E$;

(c) $P_f(1E,F) \subseteq P_f(1E,F)$ for every Banach space $E$.

In particular, in the unital case, we have the following proposition.

**Proposition 1.5.** Let $F$ be a unital Banach algebra. Then the following are equivalent:

(a) $F$ is a finite-dimensional space;

(b) $P_f(nE,F) = P_f(nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space $E$;

(c) $P_f(1E,F) = P_f(1E,F)$ for every Banach space $E$.

2. Proofs

**Lemma 2.1.** Let $n$ be any positive integer and let $x_1, \ldots, x_n$ be $n$-real variables. Then

\[
\prod_{i=1}^{n} x_i = \frac{1}{2^n n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k x_k \right)^n
\]

(2.1)

holds.

**Proof.** For each $m$ with $0 \leq m \leq n$, let

\[
P_m(x_1, \ldots, x_n) = \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k x_k \right)^m
\]

(2.2)
Then we have \( P_m(0, x_2, \ldots, x_n) = P_m(x_1, 0, \ldots, x_n) = \cdots = P_m(x_1, \ldots, x_{n-1}, 0) = 0 \). Indeed since

\[
P_m(x_1, \ldots, x_n) = \sum_{\varepsilon_2, \ldots, \varepsilon_n = \pm 1} \varepsilon_2 \cdot \varepsilon_n (x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n)^m
\]

\( - \sum_{\varepsilon_2, \ldots, \varepsilon_n = \pm 1} \varepsilon_2 \cdot \varepsilon_n (-x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n)^m, \) \hspace{1cm} (2.3)

it follows that \( P_m(0, x_2, \ldots, x_n) = 0 \). Similarly,

\[
P_m(x_1, 0, \ldots, x_n) = \cdots = P_m(x_1, \ldots, x_{n-1}, 0) = 0. \) \hspace{1cm} (2.4)

Therefore, we have

\[
P_m(x_1, \ldots, x_n) = 0, \] \hspace{1cm} (2.5)

for each \( m = 0, 1, 2, \ldots, n-1 \) and

\[
P_n(x_1, \ldots, x_n) = K_n \prod_{i=1}^{n} x_i, \] \hspace{1cm} (2.6)

for some constant \( K_n \), because \( P_m(x_1, \ldots, x_n) \) is \( m \)-homogeneous for \( x_1, \ldots, x_n \). Hence we only show that \( K_n = 2^n n! \). Note that

\[
K_n = P_n(1, \ldots, 1) = \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k \right)^n. \] \hspace{1cm} (2.7)

Then \( K_1 = 2 \). Now, for each \( m \) with \( 0 \leq m \leq n \), let \( \alpha_m = \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k \right)^m \). Then by (2.5) and (2.6), we have \( \alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0 \) and \( \alpha_n = K_n \). Hence,

\[
K_{n+1} = \sum_{\varepsilon_1, \ldots, \varepsilon_{n+1} = \pm 1} \varepsilon_1 \cdot \varepsilon_{n+1} \left( \sum_{k=1}^{n+1} \varepsilon_k \right)^{n+1}
\]

\[
= \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k + 1 \right)^{n+1} - \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k - 1 \right)^{n+1}
\]

\[
= \sum_{m=0}^{n+1} \binom{n+1}{m} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k \right)^{m}
\]

\[
- \sum_{m=0}^{n+1} \binom{n+1}{m} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k \right)^{m}
\]

\[
= \sum_{m=0}^{n+1} \binom{n+1}{m} (1 - (-1)^{n+1-m}) \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdot \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k \right)^{m}
\]

\[
= \sum_{m=0}^{n+1} \binom{n+1}{m} (1 - (-1)^{n+1-m}) \alpha_m
\]

\[
= \binom{n+1}{n} (1 - (-1)^{n+1}) K_n
\]

\[
= 2(n+1) K_n,
\]

so that we have \( K_n = 2^n n! \) \((n = 1, 2, \ldots)\) inductively. \( \square \)
**Proof of Proposition 1.4.** (a)⇒(b). Let \( \{u_1, \ldots, u_N\} \) be a basis of \( F \) and \( g_1, \ldots, g_N \) the corresponding coordinate functionals, that is, \( g_i(u_j) = \delta_{ij} \) \((i, j = 1, \ldots, N)\). Let \( P \in P_f(\ell E, F) \). Then we can write \( P(x) = \sum_{i=1}^{\ell} (T_i(x))^n \) \((x \in E)\) for some \( T_1, \ldots, T_\ell \in B(E, F) \). Let

\[
f_{ij}(x) = g_j(T_i(x)) \quad (x \in E),
\]

for each \( i = 1, \ldots, \ell, j = 1, \ldots, N \). Then we have \( T_i(x) = \sum_{j=1}^{N} f_{ij}(x) u_j \) \((x \in E, i = 1, \ldots, \ell)\), and hence by Lemma 2.1,

\[
P(x) = \sum_{i=1}^{\ell} \left( \sum_{j=1}^{N} f_{ij}(x) u_j \right)^n
\]

\[
= \sum_{i=1}^{\ell} \prod_{j_1=1}^{N} \cdots \prod_{j_n=1}^{N} f_{i,j_1}(x) \cdots f_{i,j_n}(x) u_{j_1} \cdots u_{j_n}
\]

\[
= \sum_{i=1}^{\ell} \prod_{j_1=1}^{N} \cdots \prod_{j_n=1}^{N} \frac{1}{K_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left( \sum_{k=1}^{n} \varepsilon_k f_{i,j_k}(x) \right)^n u_{j_1} \cdots u_{j_n}
\]

\[
= \sum_{i=1}^{\ell} \prod_{j_1=1}^{N} \cdots \prod_{j_n=1}^{N} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left( f_{i,j_1, \ldots, j_n, \varepsilon_1, \ldots, \varepsilon_n}(x) \right)^n b_{j_1, \ldots, j_n, \varepsilon_1, \ldots, \varepsilon_n}
\]

for each \( x \in E \), where \( f_{i,j_1, \ldots, j_n, \varepsilon_1, \ldots, \varepsilon_n} = \varepsilon_1 f_{i,j_1} + \cdots + \varepsilon_n f_{i,j_n} \in E^* \) and \( b_{j_1, \ldots, j_n, \varepsilon_1, \ldots, \varepsilon_n} = (1/K_n)\varepsilon_1 \cdots \varepsilon_n u_{j_1} \cdots u_{j_n} \in F \). Therefore we have \( P \in P_f(n E, F) \).

(b)⇒(c). This is trivial.

(c)⇒(a). Suppose that \( P_f(1 E, F) \subseteq P_f(1 E, F) \) for every Banach space \( E \). Note that \( P_f(1 F, F) = \{ T \in B(F, F) : \dim T(F) < \infty \} \) and \( P_f(1 F, F) = B(F, F) \). Then by hypothesis, the identity map of \( F \) onto itself is finite dimensional and so is \( F \). \qed

**Proof of Proposition 1.5.** This follows immediately from Propositions 1.1 and 1.4. \qed

**References**