NOTES ON WHITEHEAD SPACE OF AN ALGEBRA

M. ARIAN-NEJAD

Received 2 August 2001

Let $R$ be a ring, and denote by $[R,R]$ the group generated additively by the additive commutators of $R$. When $R_n = M_n(R)$ (the ring of $n \times n$ matrices over $R$), it is shown that $[R_n,R_n]$ is the kernel of the regular trace function modulo $[R,R]$. Then considering $R$ as a simple left Artinian $F$-central algebra which is algebraic over $F$ with $\text{Char} F = 0$, it is shown that $R$ can decompose over $[R,R]$, as $R = Fx + [R,R]$, for a fixed element $x \in R$. The space $R/[R,R]$ over $F$ is known as the Whitehead space of $R$. When $R$ is a semisimple central $F$-algebra, the dimension of its Whitehead space reveals the number of simple components of $R$. More precisely, we show that when $R$ is algebraic over $F$ and $\text{Char} F = 0$, then the number of simple components of $R$ is greater than or equal to $\dim_F R/[R,R]$, and when $R$ is finite dimensional over $F$ or is locally finite over $F$ in the case of $\text{Char} F = 0$, then the number of simple components of $R$ is equal to $\dim_F R/[R,R]$.

2000 Mathematics Subject Classification: 12E15, 16K40.

1. Introduction. Additive commutator elements of a ring $R$ and the groups and structures they make have a great role in the general specification of a ring, and their study is one of the approaches to recognize rings in noncommutative ring theory [2, 3, 4, 5]. The reason is clear, they have covered the secrets of noncommutative behaviour of the structure. In recent years, these elements are returned once again under a full consideration, and a lot of wonderful works has been done on them [1, 10, 11, 12, 13]. Our study here is also among these studies, and it reveals some of bilateral relations between substructure given by additive commutators (the additive commutator group $[R,R]$, the additive Whitehead group, and the space $R/[R,R]$) and some characteristics of the ring. In what follows let $R$ be a ring. By $[R,R]$ we denote the group generated additively by the additive commutators of $R$. Following [2], the additive group $R/[R,R]$ is called the additive Whitehead group of $R$. This group is an $F$-vector space when $R$ is a central $F$-algebra, and is called the Whitehead space of $R$.

2. Results. Our first result is about the additive commutator subgroup of a matrix ring over a given ring.

**Proposition 2.1.** Let $R$ be a unitary ring and let $R_n = M_n(R)$ be the ring of $n \times n$ matrices over $R$. Consider the regular trace function on $R_n$, as $\text{tr} : R_n \to R$, then

$$[R_n,R_n] = \{ A \in R_n \mid \text{tr}(A) \in [R,R] \}.$$  \hspace{1cm} (2.1)

**Proof.** The inclusion "$\subseteq$" follows by the fact that $\text{tr}(AB - BA) \in [R,R]$. In order to show the reverse inclusion, let $\{E_{ij}\}$ be the matrix units and note that if $i \neq j$, we have $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii} \in [R_n,R_n]$ and $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij} \in [R_n,R_n]$. For any
$A = (a_{ij}) \in R_n$, we have the following congruence:

$$A = \Sigma a_{ij}E_{ij} \equiv \Sigma a_{ii}E_{ii} \equiv \Sigma a_{ii}E_{i11} \pmod{[R_n, R_n]}.$$  \hspace{1cm} (2.2)

In particular, if $\text{tr}(A) \in [R, R]$, then $A \in [R, R_n]$. \hfill \square

**Corollary 2.2.** Consider the trace function on $R_n$ module of $[R, R]$. Clearly the group isomorphism $R_n/[R_n, R_n] \cong R/[R, R]$ can be derived.

**Theorem 2.3.** Let $R$ be a left Artinian central simple $F$-algebra which is algebraic over $F$ with $\text{Char} F = 0$. Then $R$ decomposes over $[R, R]$ as $R = Fx + [R, R]$, for a fixed $x \in R$.

**Proof.** By Wedderburn-Artin theorem, $R = M_n(D)$ for a division ring $D$ and suitable $n \in \mathbb{N}$. We divide our proof into two parts.

(i) Let $n = 1$, in other words let $R = M_1(D) = D$ be a division ring. Let $a \in R$ and let $f(t) = t^n + b_1 t^{n-1} + \cdots + b_r$ be the minimal polynomial of $a$ over $F$, where $b_i \in F$, $i = 1, 2, \ldots, r$ and $r = \dim_F F(a)$. By the Wedderburn theorem [9, page 265], $f(t)$ splits completely in $R[t]$, this means that there exists $c_i \in R^* = D - \{0\}$, $i = 1, 2, \ldots, r - 1$, such that $f(t) = (t - a)(t - c_1ac_1^{-1}) \cdots (t - cr_{-1}ac_{r-1}^{-1})$. Then we have

$$\text{Tr}_{F(a)/F}(a) = a + c_1ac_1^{-1} + c_2ac_2^{-1} + \cdots + cr_{-1}ac_{r-1}^{-1} = ra + (c_1ac_1^{-1} - a) + \cdots + (cr_{-1}ac_{r-1}^{-1} - a) \hspace{1cm} (2.3)$$

$$= ra + (c_1(ac_1^{-1} - ac_1^{-1})c_1 + \cdots + (cr_{-1}(ac_{r-1}^{-1} - ac_{r-1}^{-1})c_{r-1})$$

$$= ra + d_1 + d_2 + \cdots + d_{r-1} = ra + d,$$

where $d_1, \ldots, d_{r-1}, d \in [R, R]$. This simply yields $a \in F + [R, R]$ which imply that $R = F + [R, R]$, $x = 1$.

(ii) Let $n \in \mathbb{N}$ be an arbitrary positive integer. We have $R = M_n(D)$, where $D$ is a division ring. By (i), $D = F + [D, D]$, so

$$R = M_n(D) = M_n(F + [D, D]) = M_n(F) + M_n([D, D]) \subseteq M_n(F) + [R, R] \subseteq R. \hspace{1cm} (2.4)$$

This implies that $R = M_n(F) + [R, R]$. By this formula, given $A \in R$, there exist $B \in M_n(F)$ and $C \in [R, R]$ such that $A = B + C$, hence $A = (B - (\text{tr}B/n)I) + (\text{tr}B/n)I + C$, where $I$ is the identity matrix of size $n$. By Proposition 2.1, $(B - (\text{tr}B/n)I) \in [R, R]$, and $A = (\text{tr}B/n)I + ((B - (\text{tr}/n)I) + C$, consequently

$$R = FI + [R, R], \hspace{1cm} x = I. \hspace{1cm} (2.5)$$

To see a different statements and initial ideas of these theorems we refer the reader to [1, 2]. Also a multiplicative version of Theorem 2.3 could be found in [11].

Now, we are going to state our main result, which is about the Whitehead space of a semisimple ring. This theorem is a generalization of a nice theorem due to $R. \text{Brauer}$ [8, page 130].
**Theorem 2.4.** Let \( R \) be a left Artinian semisimple central \( F \)-algebra and let \( k \) be the number of left simple components of \( R \). Then,

1. if \( R \) is algebraic over \( F \) and \( \text{Char} F = 0 \), then \( k \geq \dim_F R/[R,R] \);
2. if \( R \) is finite dimensional over \( F \), or is locally finite over \( F \), and \( \text{Char} F = 0 \), then \( k = \dim_F R/[R,R] \).

**Proof.** Consider the following chain of functions:

\[
R \xrightarrow{f_1} M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k) \xrightarrow{f_2} D_1/[D_1,D_1] \times \cdots \times D_k/[D_k,D_k],
\]

where \( f_1 \) is the isomorphism given by the Wedderburn-Artin theorem for the decomposition of a semisimple left Artinian ring into a direct product of simple rings [6, 14], and \( f_2 \) is the \( F \)-algebra homomorphisms, by considering component-wise the trace function on \( M_{n_i}(D_i) \mod[D_i,D_i], i = 1, \ldots, k \).

By Proposition 2.1 we have, \( \ker(f_2 \circ f_1) = [R,R] \), noting that \([R,R] \cong [R_1,R_1] \times \cdots \times [R_k,R_k] \), where \( R_{n_i} = M_{n_i}(D_i), i = 1, \ldots, k \). Therefore the following \( F \)-isomorphism holds:

\[
R/[R,R] \cong D_1/[D_1,D_1] \times \cdots \times D_k/[D_k,D_k].
\]

It remains to compute the dimension of Whitehead space of a division ring in the two cases (i) and (ii) above.

First let \( D \) be algebraic over \( F \) and \( \text{Char} F = 0 \). We show that any two elements \( \bar{a}, \bar{b} \in D/[D,D] \) are linearly dependent. By Theorem 2.3, there exist elements \( \alpha, \beta \in F \) and \( d_1, d_2 \in [D,D] \), such that \( a = \alpha + d_1 \) and \( b = \beta + d_2 \). In other words, \( \beta \bar{a} - \alpha \bar{b} = 0 \) in \( D/[D,D] \). Hence in this case \( \dim_F D/[D,D] \leq 1 \).

Now let \( D \) be finite dimensional \( F \)-central algebra. Let \( RT_{D/F} : D \to F \) be the reduced trace function which is surjective by [7, page 148]. Furthermore, by a theorem of Amitsur and Rowen [5, page 171] its kernel is equal to \([D,D] \) and so it is a hyperplane over \( F \), in this case \( \dim_F D/[D,D] = 1 \).

As a latter case let \( D \) be a locally finite division ring over it’s center \( F \) and \( \text{Char} F = 0 \). Now consider the function \( TR : D \to F \) defined by

\[
TR(x) = \frac{1}{\deg_F(x)} \text{Tr}_{F(x)/F}(x),
\]

we show that this function is an \( F \)-linear surjective map, whose kernel is \([D,D] \). The claim then is clear.

First note that in this case \( 1 \notin [D,D] \), for if \( 1 \in [D,D] \), then there exist some \( x_i \)'s and \( y_i \)'s in \( D \), such that \( 1 = \sum (x_i y_i - y_i x_i) \). Let \( D_1 \) be the division ring generated by \( F \) together with \( x_i \)'s and \( y_i \)'s. Taking the reduced trace of \( D_1 \) over its centre of both sides of \( 1 = \sum(x_i y_i - y_i x_i) \), we get a contradicting result. Therefore \([D,D] \cap F = \{0\} \). Now, by considering the trace formula (given in the proof of Theorem 2.3) for elements \( a, b \) and \( \lambda a + b \) (\( \lambda \in F \)) in \( D \), it is readily verified that

\[
\frac{1}{r} \text{Tr}(\lambda a + b) = \frac{\lambda}{n} \text{Tr}(a) + \frac{1}{m} \text{Tr}(b),
\]

where \( r = \text{deg}_F(x) \).
where \( r, n, \) and \( m \) are degrees of \( \lambda a + b, a \) and \( b. \) So \( TR \) is \( F \)-linear. The surjectivity is clear. In order to specify the kernel of \( TR, \) consider the trace formula for elements of \([D,D]\). Suppose that \( a \in [D,D] \). Now, we have \( \text{Tr}_{F(a)/F}(a) = na + d \in [D,D] \cap F, \) where \( n \) is the degree of \( a \) over \( F \) and \( d \in [D,D] \). Therefore \( TR(a) = 0. \) By the same argument we can see that if \( TR(a) = 0, \) then \( a \in [D,D] \).

**Acknowledgment.** The author is indebted in part to the Research Council of the University of Zanjan for support.

**References**


M. ARIAN-NEJAD: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZANJAN, ZANJAN, IRAN

E-mail address: arian@mail.znu.ac.ir