We get a new \( \mathbb{Z} \)-graded Witt type simple Lie algebra using a generalized polynomial ring which is the radical extension of the polynomial ring \( F[x] \) with the exponential function \( e^x \).

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1. Introduction. Let \( F \) be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, \( \mathbb{Z}_+ \) and \( \mathbb{Z} \) denote the nonnegative integers and the integers, respectively. Let \( F[x] \) be the polynomial ring in indeterminate \( x \). Let \( F(x) = \{ f(x)/g(x) \mid f(x), g(x) \in F[x], g(x) \neq 0 \} \) be the field of rational functions in one variable. We define the \( F \)-algebra \( V_{\sqrt{m},e} \) spanned by

\[
\left\{ e^{dx} f_{a_{1}/b_{1}}^{1} \cdots f_{a_{m}/b_{m}}^{m} x^{t} \mid d,a_{1},\ldots,a_{m},t \in \mathbb{Z}, f_{i} \neq x, (a_{1},b_{1}) = 1,\ldots,(a_{m},b_{m}) = 1, 1 \leq i \leq m \right\},
\]

where \( b_{1},\ldots,b_{m} \) are fixed nonnegative integers, and \( (a_{i},b_{i}) = 1, 1 \leq i \leq m \), means that \( a_{i} \) and \( b_{i} \) are relatively primes, and \( f_{1},\ldots,f_{n} \) are the fixed relatively prime polynomials in \( F[x] \). The \( F \)-subalgebra \( V_{\sqrt{m},e}^{+} \) of \( V_{\sqrt{m},e} \) is spanned by

\[
\left\{ e^{dx} f_{a_{1}/b_{1}}^{1} \cdots f_{a_{m}/b_{m}}^{m} x^{t} \mid d,a_{1},\ldots,a_{m} \in \mathbb{Z}, t \in \mathbb{Z}_+, f_{i} \neq x, (a_{1},b_{1}) = 1,\ldots,(a_{m},b_{m}) = 1, 1 \leq i \leq m \right\}.
\]

Let \( W_{\sqrt{m},e}(\partial) \) be the vector space over \( F \) with elements \( \{ f\partial \mid f \in V_{\sqrt{m},e} \} \) and the standard basis \( \{ e^{dx} f_{a_{1}/b_{1}}^{1} \cdots f_{a_{m}/b_{m}}^{m} x^{t}\partial \mid e^{dx} f_{a_{1}/b_{1}}^{1} \cdots f_{a_{m}/b_{m}}^{m} x^{t}\partial \in W_{\sqrt{m},e} \} \). Define a Lie bracket on \( W_{\sqrt{m},e}(\partial) \) as follows:

\[
[f\partial,g\partial] = f(\partial(g))\partial - g(\partial(f))\partial, \quad f,g \in V_{\sqrt{m},e}.
\]

It is easy to check that (1.3) defines a Lie algebra \( W_{\sqrt{m},e}(\partial) \) with the underlying vector space \( W_{\sqrt{m},e}(\partial) \) (see also [1, 3, 5]). Similarly, we define the Lie subalgebra \( W_{\sqrt{m},e}^{+}(\partial) \) of \( W_{\sqrt{m},e}(\partial) \) using the \( F \)-algebra \( V_{\sqrt{m},e}^{+} \) instead of \( V_{\sqrt{m},e} \).

The Lie algebra \( W_{\sqrt{m},e}(\partial) \) has a natural \( \mathbb{Z} \)-gradation as follows:

\[
W_{\sqrt{m},e}(\partial) = \bigoplus_{d \in \mathbb{Z}} W_{\sqrt{m},e}^{d},
\]

where \( W_{\sqrt{m},e}^{d} \) is the subspace of the Lie algebra \( W_{\sqrt{m},e}(\partial) \) generated by elements of the form \( e^{dx} f_{a_{1}/b_{1}}^{1} \cdots f_{a_{m}/b_{m}}^{m} x^{t}\partial \mid f_{1},\ldots,f_{n} \in F[x], a_{1},\ldots,a_{m},t \in \mathbb{Z}, m \in \mathbb{Z}_+ \). We call the subspace \( W_{\sqrt{m},e}^{d} \) the \( d \)-homogeneous component of \( W_{\sqrt{m},e}(\partial) \).
We decompose the \(d\)-homogeneous component \(W_{d,\sqrt{m},e}^d\) as follows:

\[
W_{d,\sqrt{m},e}^d = \bigoplus_{s_1,\ldots,s_m \in \mathbb{Z}} W_{d,s_1,\ldots,s_m},
\]

where \(W_{d,s_1,\ldots,s_m}\) is the subspace of \(W_{d,\sqrt{m},e}^d\) spanned by

\[
\{ e^{dx} f_1^{s_1/b_1} \cdots f_m^{s_m/b_m} x^q \partial \mid q \in \mathbb{Z} \}.
\]

Note that \(W_{(0,0,\ldots,0)}\) is the Witt algebra \(W(1)\) as defined in [3].

The two radical-homogeneous components \(W_{d,a_1,\ldots,a_m}\) and \(W_{d,r_1,\ldots,r_m}\) are equivalent if \(a_1 - r_1, \ldots, a_m - r_m \in \mathbb{Z}\). This defines an equivalence relation on \(W_{d,\sqrt{m},e}^d\). Thus we note that the equivalent class of \(W_{d,a_1,\ldots,a_m}\) depends only on \(a_1, \ldots, a_m\). From now on \(W_{d,a_1,\ldots,a_m}\) will represent the radical homogeneous equivalent class of \(W_{d,a_1,\ldots,a_m}\) without ambiguity. It is possible to choose the minimal positive integers \(a_1, \ldots, a_m\) for the radical homogeneous equivalent component \(W_{d,a_1,\ldots,a_m}\).

We give the lexicographic order on all the radical homogeneous equivalent components \(W_{d,a_1,\ldots,a_m}\) using \(\mathbb{Z} \times \mathbb{Z}^m_+\).

The radical equivalent homogeneous component \(W_{d,\sqrt{m},e}^d\) can be written as follows:

\[
W_{d,\sqrt{m},e}^d = \sum_{(a_1,\ldots,a_m) \in \mathbb{Z}^m} W_{d,a_1,\ldots,a_m}.
\]

Thus for any element \(l \in W_{\sqrt{m},e}(\partial)\), \(l\) can be written uniquely as follows:

\[
l = \sum_{(d,a_1,\ldots,a_m) \in \mathbb{Z} \times \mathbb{Z}^m} l_{(d,a_1,\ldots,a_m)}.\]

For any such element \(l \in W_{\sqrt{m},e}(\partial)\), \(H(l)\) is defined as the number of different homogeneous components of \(l\) as in (1.4), and \(L_d(l)\) as the number of nonequivalent radical \(d\)-homogeneous components of \(l\) in (1.8). For each basis element \(e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial\) of \(W_{\sqrt{m},e}(\partial)\) (or \(W_{+,\sqrt{m},e}(\partial)\)), define \(\deg_{Lie}(e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial) = t\). Since every element \(l\) of \(W_{\sqrt{m},e}(\partial)\) is the sum of the standard basis element, we may define \(\deg_{Lie}(l)\) as the highest power of each basis element of \(l\). Note that the Lie algebra \(W_{\sqrt{m},e}(\partial)\) is self-centralized, that is, the centralizer \(C_I(W_{\sqrt{m},e}(\partial))\) of every element \(l\) in \(W_{\sqrt{m},e}(\partial)\) is one dimensional [1]. We find the solution of

\[
1^{1/3} = y
\]

in \(\mathbb{Z}_7\). Equation (1.9) implies that

\[
1 \equiv y^3 \mod 7.
\]

The solutions of (1.10) are 1, 2, or 4. Thus \(1^{1/3} = 1, 2, \text{ or } 4 \mod 7\). Thus the radical number in \(\mathbb{Z}_p\) is not uniquely determined generally. So we may not consider the Lie algebras in this paper over a field of characteristic \(p\) differently from the Lie algebras in [2, 3, 4]. It is easy to prove that the Lie algebra \(W_{(0,\ldots,0)}\) is simple [3].
2. Main results. We need several lemmas for Theorem 2.5.

**Lemma 2.1.** For any element \( l \) in the \((d,a_1,\ldots,a_m)\)-radical-homogeneous component of \( W_{\sqrt m}(\partial) \), and for any element \( l_1 \in W_{(0,0,\ldots,0)} \), \([l,l_1]\) is an element in the \((d,a_1,\ldots,a_m)\) -radical homogeneous equivalent component.

The proof of Lemma 2.1 is straightforward.

**Lemma 2.2.** A Lie ideal \( I \) of \( W_{\sqrt m,e}(\partial) \) which contains \( \partial \) is \( W_{\sqrt m,e}(\partial) \).

**Proof.** Let \( I \) be the ideal in the lemma. The Lie subalgebra which has the standard basis \( \{x^i\partial \mid i \in \mathbb{Z}_+\} \) is simple. Let \( I \) be any ideal of \( W_{\sqrt m,e}(\partial) \) which contains \( \partial \). Then for any \( f\partial \in W_{\sqrt m,e}(\partial) \),

\[
[x\partial,f\partial] = x\partial(f)\partial - f\partial \in I. \tag{2.1}
\]

On the other hand,

\[
[\partial,xf\partial] = fx\partial(f)\partial + x\partial(f)\partial \in I. \tag{2.2}
\]

Thus by subtracting (2.2) from (2.1) we get \( 2f\partial \in I \). Therefore, we have proven the lemma, since \( I \cap W_{(0,0,\ldots,0)} \) contains nonzero elements and so \( I \supset W_{(0,0,\ldots,0)} \).

**Lemma 2.3.** A Lie ideal \( I \) of \( W_{\sqrt m,e}(\partial) \) which contains a nonzero element in \( W_{(d,a_1,\ldots,a_m)} \) is \( W_{\sqrt m,e}(\partial) \), for a fixed \((d,a_1,\ldots,a_m) \in \mathbb{Z} \times \mathbb{Z}_+\).

**Proof.** Let \( I \) be any nonzero Lie ideal of \( W_{\sqrt m,e}(\partial) \). For any nonzero element \( l \in I \), there is an element \( x^s\partial \), \( s \gg 0 \), such that \([x^s\partial,l]\) is the sum of elements in \( W_{\sqrt m,e}(\partial) \) with \( \deg_{\text{Lie}}([x^s\partial,l]) > 0 \).

**Theorem 2.5.** The Lie algebra \( W_{\sqrt m,e}(\partial) \) is simple.

**Proof.** Let \( I \) be a nonzero Lie ideal of \( W_{\sqrt m,e}(\partial) \). By Lemma 2.4, we may assume that \( l \) has polynomial terms with positive powers for each basis element of \( l \). We prove this theorem in several steps.

**Step 1.** If \( l \) is in the 0-homogeneous component, then the theorem holds. We prove this step, by induction on the number \( L_0(l) \) of nonequivalent radical-homogeneous components of the element \( l \) of \( I \). If \( L_0(l) = 1 \) and \( l \in W_{(0,0,\ldots,0)} \), then the theorem holds by Lemmas 2.2, 2.3, and the fact that \( W_{(0,0,\ldots,0)} \) is simple.
Assume that \( l \in W_{(0,0\ldots,0,a_r\ldots,a_m)} \) with \( a_r \neq 0 \). If we take an element \( f_1^{h_r/k_r} \cdots f_n^{h_m/k_m} x^{h_{m+1}} \partial \) such that \( h_r \gg k_r, \ldots, h_n \gg k_r \) and \((h_r + k_r)/k_r \in \mathbb{Z}_+, \ldots, (h_m + k_m)/k_m \in \mathbb{Z}_+\), then we have \( l_1 = [f_1^{h_r/k_r} \cdots f_n^{h_m/k_m} x^{h_{m+1}} \partial, l] \neq 0 \). This implies that \( l_1 \) is in \( W(0,0,\ldots,0) \). Thus we have proven the theorem by Lemma 2.2.

By induction, we may assume that the theorem holds for \( l \in I \) such that \( L_0(l) = k \), for some fixed nonnegative integer \( k > 1 \). Assume that \( L_0(l) = k + 1 \). If \( l \) has a \( W_{(0,0\ldots,0)} \) radical-homogeneous equivalent component, we take \( l_2 \in W_{(0,0\ldots,0)} \) such that \([l, l_2]\) can be written as follows: \([l, l_2] = l_3 + l_4\) where \( l_3 \) is a sum of nonzero radical-homogeneous components, and \( l_4 = f \partial \) with \( f \in F[x] \). Thus we have the nonzero element

\[
\partial, [\cdots, [\partial, l] \cdots] = l_2 \in I
\]

(2.3)

which has no terms in the homogeneous equivalent component \( W_{(0,0\ldots,0)} \), where we applied Lie brackets until \( l_2 \) has no terms in the radical homogeneous equivalent component \( W_{(0,0\ldots,0)} \). Then \( l_2 \in I \) such that \( H(l_2) \leq k \). Therefore, we have proven the theorem by Lemmas 2.2, 2.3, and induction. If \( l \) has no terms in the radical homogeneous equivalent component \((0,0\ldots,0)\), then \( l \) has a term in the radical homogeneous equivalent component \( W_{(0,a_1\ldots,a_m)} \). Take an element \( l_3 = f_1^{c_1/p_1} \cdots f_m^{c_m/p_m} x^{c_{m+1}} \partial \) such that \( c_1, \ldots, c_{m+1} \) are sufficiently large positive integers such that \( c_1 + a_1 \in \mathbb{Z} \cdots c_m + a_m \in \mathbb{Z} \), and which is in a radical homogeneous equivalent component \( W_{(0,a_1\ldots,a_m)} \). Then \([l_3, l]\) is nonzero and which has a term in the radical homogeneous equivalent component \( W_{(0,0\ldots,0)} \). So in this case we have proven the theorem by induction.

**Step 2.** Assume that \( l \) is in the \( d \)-homogeneous component such that \( 0 \neq d \) and \( L_0(l) = 1 \), then the theorem holds. By taking \( e^{-dx} x^t \partial \), we have \( 0 \neq [e^{-dx} x^t \partial, l] \in W_{(0,0\ldots,0)} \) by taking a sufficiently large positive integer \( t \). Thus we have proven the theorem by Step 1.

**Step 3.** If \( l \) is the sum of \((k - 1)\) nonzero homogeneous components and 0-homogeneous component, then the theorem holds. We prove the theorem by induction on the number of distinct homogeneous components by Steps 1 and 2. Assume that we have proven the theorem when \( l \) has \((k - 1)\) radical-homogeneous components. Assume that \( l \) has terms in \( W_{(0,0\ldots,0)} \). By Step 1, we have an element \( l_1 \in I \), such that \( l_1 = l_2 + f \partial \), where \( l_2 \) has \((k - 1)\) homogeneous components and \( f \in F[x] \). Then \( 0 \neq \partial, [\cdots, [\partial, l_1] \cdots] \in I \) has \((k - 1)\) homogeneous components, where we applied the Lie bracket until it has no terms in \( W_{(0,0\ldots,0)} \). Therefore, we have proven the theorem by induction.

Assume that \( l \) has a \((k)\) homogeneous equivalent components. We may assume \( l \) has the terms which is in \( 0 \neq d \)-homogeneous component. By taking a sufficiently large positive integer \( r \), we have \([e^{-dx} x^r \partial, l] \neq 0 \) and it has \((k)\) homogeneous components with a term in the radical-homogeneous component \( W_{(0,0\ldots,0)} \). Therefore, we have proven the theorem by Step 3. \( \square \)

**Corollary 2.6.** The Lie algebra \( W_{\sqrt{m},e}^+(\partial) \) is simple.

**Proof.** It is straightforward from Theorem 2.5 without using Lemma 2.4. \( \square \)

**Corollary 2.7.** The Lie subalgebra \( W_{\sqrt{m},e}^0 \) of \( W_{\sqrt{m},e}^+(\partial) \) is simple. \( \square \)
Proof. It is straightforward from Step 1 of Theorem 2.5.

Proposition 2.8. For any nonzero Lie automorphism \( \theta \) of \( W_{\sqrt{\text{mc}}} + \partial \), \( \theta(\partial) = \partial \) holds.

Proof. It is straightforward from the relation \( \theta([\partial, x\partial]) = \theta(\partial) \) and the fact that \( W_{\sqrt{\text{mc}}} + \partial \) is self-centralized and \( \mathbb{Z} \)-graded.

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References


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