METHOD OF REPLACING THE VARIABLES FOR GENERALIZED 
SYMMETRY OF THE D’ALEMBERT EQUATION

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We show that by the generalized understanding of symmetry, the D’Alembert equation for 
one component field is invariant with respect to arbitrary reversible coordinate transfor-
mations.

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1. Introduction. Symmetries play an important role in particle physics and quan-
tum field theory [1], nuclear physics [11], and mathematical physics [5]. Some recep-
tions are proposed for finding the symmetries, for example, the method of replacing 
the variables [13], the Lie algorithm [5], and the theoretical-algebraic approach [9]. The 
purpose of this work is the generalization of the method of replacing the variables. 
We start from the following definition of symmetry.

2. Main results

DEFINITION 2.1. Let a differential equation \( \hat{L}' \phi'(x') = 0 \) be given. By symmetry 
of this equation with respect to the variables replacement \( x' = x'(x) \), \( \phi' = \phi'(\Phi \phi) \) we will understand the compatibility of the engaging equations system \( \hat{A} \phi'(\Phi \phi) = 0 \), 
\( \hat{L} \phi(x) = 0 \), where \( \hat{A} \phi'(\Phi \phi) = 0 \) is obtained from the initial equation by replacing 
the variables, \( \hat{L}' = \hat{L} \), \( \Phi(x) \) is some weight function. If the equation \( \hat{A} \phi'(\Phi \phi) = 0 \) can be 
transformed into the form \( \hat{L}(\Psi \phi) = 0 \), the symmetry will be named the standard Lie 
symmetry, otherwise it will be named generalized symmetry.

2.1. Application of Definition 2.1. The elements of Definition 2.1 were used to 
study the Maxwell equations symmetries [6, 7, 8]. In this paper, we apply this defi-
nition for investigation of symmetries of the one-component D’Alembert equation
\[
\square' \phi'(x') = \frac{\partial^2 \phi'}{\partial x'^2_1} + \frac{\partial^2 \phi'}{\partial x'^2_2} + \frac{\partial^2 \phi'}{\partial x'^2_3} + \frac{\partial^2 \phi'}{\partial x'^2_4} = 0. \tag{2.1}
\]

We introduce the arbitrary reversible coordinate transformations \( x' = x'(x) \) and the 
transformation of the field variable \( \phi' = \phi'(\Phi \phi) \), where \( \Phi(x) \) is some unknown func-
tion, also we take into account
\[
\frac{\partial \phi'}{\partial x'_i} = \sum_j \frac{\partial \phi'}{\partial x'_j} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x'_i},
\]
\[
\frac{\partial^2 \phi'}{\partial x'^2_i} = \sum_j \frac{\partial^2 \phi}{\partial x'_j} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x'_i} + \sum_{jk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \frac{\partial \phi}{\partial x'_i} \frac{\partial \phi}{\partial x'_i} + \sum_{jk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \frac{\partial \phi}{\partial x'_i} \frac{\partial \phi}{\partial x'_i} \frac{\partial \phi}{\partial x'_i}. \tag{2.2}
\]
where $\xi = \Phi \phi$. After replacing the variables we find that the equation $\Box' \phi' = 0$ transforms into itself, if the system of the engaging equations is fulfilled

$$
\sum_i \sum_j \frac{\partial^2 x_j}{\partial x_i^2} \frac{\partial \phi'}{\partial \xi} \frac{\partial \phi}{\partial x_j} + \sum_i \left( \frac{\partial x_j}{\partial x_i} \right)^2 \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi \phi}{\partial x_j^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i} \frac{\partial x_k}{\partial x_i} \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi \phi}{\partial x_j \partial x_k} + \sum_i \sum_{j<k} \sum_k \left( \frac{\partial x_j}{\partial x_i} \right)^2 \frac{\partial^2 \phi'}{\partial \xi^2} \left( \frac{\partial \Phi \phi}{\partial x_j} \right)^2 = 0; \quad \Box \phi = 0.
$$

(2.3)

Here $x = (x_1, x_2, x_3, x_4)$, $x_4 = ict$, where $c$ is the speed of light and $t$ is the time. We substitute the solution of the D’Alembert equation $\phi$ into the first equation of the set (2.3). If the obtained equation has a solution, then the set (2.3) will be compatible. According to Definition 2.1 this compatibility will mean that arbitrary reversible transformations $x' = x'(x)$ are the symmetry transformations of the initial equation $\Box' \phi' = 0$. Owing to the presence of the expressions $(\partial \Phi \phi/\partial x_j)^2$ and $(\partial \Phi \phi/\partial x_j)(\partial \Phi \phi/\partial x_k)$ in the first equation of (2.3), the latter has nonlinear character. Since the analysis of nonlinear systems is difficult we suppose that

$$
\frac{\partial^2 \phi'}{\partial \xi^2} = 0.
$$

(2.4)

In this case, the nonlinear components in the set (2.3) turn to zero and the system will be linear. As a result, we find the field transformation law by integrating (2.4)

$$
\phi' = C_1 \Phi \phi + C_2 \to \phi' = \Phi \phi.
$$

(2.5)

Here we suppose for simplicity that the constants of integration are $C_1 = 1$, $C_2 = 0$. It is this law of field transformation that was used within the algorithm [7, 8]. It marks the position of the algorithm in the generalized variables replacement method. Taking into account formulae (2.4) and (2.5), we find the following form for system (2.3):

$$
\sum_j \Box' x_j \frac{\partial \Phi \phi}{\partial x_j} + \sum_i \left( \frac{\partial x_j}{\partial x_i} \right)^2 \frac{\partial^2 \phi}{\partial x_j^2} + \sum_i \sum_{j<k} 2 \frac{\partial x_j}{\partial x_i} \frac{\partial x_k}{\partial x_i} \frac{\partial^2 \Phi \phi}{\partial x_j \partial x_k} = 0; \quad \Box \phi = 0.
$$

(2.6)

Since here $\Phi(x) = \phi'(x' \to x)/\phi(x)$, where $\phi'(x')$ and $\phi(x)$ are the solutions of the D’Alembert equation, system (2.6) has a common solution and consequently is compatible. This means that the arbitrary reversible transformations of coordinates $x' = x'(x)$ are symmetry transformations for the one-component D’Alembert equation if the field is transformed with the help of the weight function $\Phi(x)$ according to the law (2.5). The form of this function depends on the D’Alembert equation solutions and the law of the coordinate transformations $x' = x'(x)$.

Next we consider the following examples.
2.1.1. Poincaré group. Let the coordinate transformations belong to the Poincaré group $P_{10}$:

$$x'_j = L_{jk} x_k + a_j,$$

(2.7)

where $L_{jk}$ is the matrix of the Lorentz group $L_0$, $a_j$ are the parameters of the translation group $T_4$. In this case, we have $\square' x_j = \sum_k L'_{jk} \square' x'_k = 0$, $\sum_i (\partial x_j / \partial x'_i) (\partial x_k / \partial x'_i) = \sum_i L'_{ij} L'_{ki} = \delta_{jk}$. The last term in the second equation of (2.6) turns to zero. The set reduces to the form

$$\square \Phi = 0; \quad \square \phi = 0.$$  (2.8)

According to Definition 2.1 this is a sign of the Lie symmetry. The weight function belongs to the set in [8]:

$$\Phi_{P_{10}}(x) = \Phi(x) \in \left\{ 1; \frac{1}{\phi(x)}; \frac{P_j \phi(x)}{\phi(x)}; \frac{M_{jk} \phi(x)}{\phi(x)}; \frac{P_j P_k \phi(x)}{\phi(x)}; \frac{P_j M_{kl} \phi(x)}{\phi(x)}; \ldots \right\},$$

(2.9)

where $P_j$, $M_{jk}$ are the generators of Poincaré group, $j, k, l = 1, 2, 3, 4$. In the space of the D’Alembert equation solutions the set defines a rule of transforming of a solution $\phi(x)$ to another solution $\phi'(x)$. The weight function $\Phi(x) = 1 \in \Phi_{P_{10}}(x)$ determines the transformational properties of the solutions $\phi' = \phi$, which means the well-known relativistic symmetry of the D’Alembert equation [4, 10].

2.1.2. Weyl group. Let the transformations of coordinates belong to the Weyl Group $W_{11}$:

$$x'_j = \rho L_{jk} x_k + a_j,$$

(2.10)

where $\rho = \text{const}$ is the parameter of the scale transformations of the group $\Delta_1$. In this case we have $\square' x_j = \rho' \sum_k L'_{jk} \square' x'_k = 0$, $\sum_i (\partial x_j / \partial x'_i) (\partial x_k / \partial x'_i) = \sum_i \rho'^2 L'_{ij} L'_{ki} = \rho'^2 \delta_{jk} = \rho^{-2} \delta_{jk}$. The set (2.6) reduces to the set (2.8) and has the solution $\Phi_{W_{11}} = C \Phi_{P_{10}}$, where $C$ = const. The weight function $\Phi(x) = C$ and the law $\phi' = C \phi$ means the well-known Weyl symmetry of the D’Alembert equation [4, 10]. Here $C = \rho^l$, where $l$ is the conformal dimension of the field $\phi(x)$ [2]. Consequently, the D’Alembert equation is $W_{11}$-invariant for the field $\phi$ with arbitrary conformal dimension $l$. This property is essential for the Voigt [13] and Umov [12].

2.1.3. Inversion group. Let the coordinate transformations belong to the inversion group $I$:

$$x'_j = -\frac{x_j}{x^2}; \quad x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2; \quad x^2 x'^2 = 1.$$  (2.11)

In this case, we have $\square' x_j = 4x'_j / x^4 = -4x_j x^2$, $\sum_i (\partial x_j / \partial x'_i) (\partial x_k / \partial x'_i) = \rho'^2 (x') \delta_{jk} = 1 / x'^4 \delta_{jk}$. The set (2.6) reduces to the set

$$-4x_j \frac{\partial \Phi \phi}{\partial x_j} + x^2 \square \Phi \phi = 0; \quad \square \phi = 0.$$  (2.12)

The substitution of $\Phi(x) = x^2 \Psi(x)$ transforms equation (2.12) for $\Phi(x)$ into the equation $\square \Psi \phi = 0$ for $\Psi(x)$. It is a sign of the Lie symmetry. The equation has the solution $\Psi = 1$. The result is $\Phi(x) = x^2$. Consequently, the field transforms according to the law $\phi' = x^2 \phi(x) = \rho^{-1} (x) \phi(x)$. This means the conformal dimension $l = -1$ of the
field \( \phi(x) \) in the case of the D’Alembert equation symmetry with respect to the inversion group \( I \) in agreement with [4, 9]. In a general case the weight function belongs to the set

\[
\Phi_1(x) = x^2 \Psi(x) \in \left\{ x^2; x^2 \frac{\phi(x)}{\phi(x)}; x^2 \frac{P_j \phi(x)}{\phi(x)}; \ldots \right\}. \tag{2.13}
\]

### 2.1.4. Special conformal group.

Let the coordinate transformations belong to the special conformal group \( C_4 \):

\[
x'_j = \frac{x_j - a_j x^2}{\sigma(x)}; \quad \sigma(x) = 1 - 2a \cdot x + a^2 x^2; \quad \sigma' = 1. \tag{2.14}
\]

In this case, we have \( \Box' x_j = 4(a_j - a^2 x_j) \sigma(x) \), \( \sum_i (\partial x_j / \partial x'_i) (\partial x_k / \partial x'_i) = \rho^2(x') \delta_{jk} = \sigma^2(x) \delta_{jk} \). The set (2.6) reduces to the set

\[
\begin{aligned}
4\sigma(x)(a_j - a^2 x_j) \frac{\partial \phi}{\partial x_j} + \sigma^2(x) \Box \phi = 0; \\
\Box \phi = 0. \tag{2.15}
\end{aligned}
\]

The substitution of \( \Phi(x) = \sigma(x) \Psi(x) \) transforms (2.15) into the equation \( \Box \Psi \phi = 0 \) which corresponds to the Lie symmetry. From this equation, we have \( \Psi = 1, \Phi(x) = \sigma(x) \). Therefore, \( \phi' = \sigma(x) \phi(x) \) and the conformal dimension of the field is \( l = -1 \) as above. Analogously to (2.13), the weight function belongs to the set

\[
\Phi_{C_4}(x) = \sigma(x) \Psi(x) \in \left\{ \sigma(x); \frac{\sigma(x) \phi(x)}{\phi(x)}; \sigma(x) \frac{P_j \phi(x)}{\phi(x)}; \sigma(x) \frac{M_{jk} \phi(x)}{\phi(x)}; \ldots \right\}. \tag{2.16}
\]

Thus, we can see that \( \phi(x) = 1/\sigma(x) \) is the solution of the D’Alembert equation. Combination of \( W_{11}, I \), and \( C_4 \) symmetries means the well-known D’Alembert equation conformal \( C_{15} \)-symmetry [4, 9, 10].

### 2.1.5. Galilei group.

Let the coordinate transformations belong to the Galilei group \( G_1 \):

\[
x'_1 = x_1 + i \beta x_4; \quad x'_2 = x_2; \quad x'_3 = x_3; \quad x'_4 = y x_4; \quad c' = yc, \tag{2.17}
\]

where \( \beta' = -\beta/y, y' = 1/y, \beta = V/c, y = (1 - 2\beta n_x + \beta^2)^{1/2} \). In this case, we have

\[
\begin{aligned}
\Box' x_j &= 0; \quad \sum_i \left( \frac{\partial x_j}{\partial x'_i} \right)^2 = 1 - \beta^2; \\
\sum_i \left( \frac{\partial x_2}{\partial x'_i} \right)^2 &= \sum_i \left( \frac{\partial x_3}{\partial x'_i} \right)^2 = 1; \quad \sum_i \left( \frac{\partial x_4}{\partial x'_i} \right)^2 = y'^2; \\
\sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_2}{\partial x'_i} &= \sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i} \frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i} \frac{\partial x_4}{\partial x'_i} = 0; \\
\sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_4}{\partial x'_i} &= i \beta' y' = -i \beta \frac{y'}{y^2};
\end{aligned} \tag{2.18}
\]

After putting these expressions into the set (2.6) we find (see [8])

\[
\Box \Phi \phi - \frac{\partial^2 \Phi \phi}{\partial x_4^2} - \left( i \frac{\partial}{\partial x_4} + \beta \frac{\partial}{\partial x_1} \right)^2 \phi \phi = \left[ \frac{(i \partial_4 + \beta \partial_1)^2}{y^2} - \triangle \right] \phi \phi = 0. \tag{2.19}
\]
In accordance with Definition 2.1 the Galilei symmetry of the D’Alembert equation is the generalized symmetry (being the conditional one [8]). The weight function belongs to the set (see [7])

\[ \Phi_{G_1}(x) = \frac{\phi'(x' - x)}{\phi(x)} \in \left\{ \frac{\phi(x')}{\phi(x)}, \frac{1}{\phi(x)}, \frac{\partial^2 \phi(x')}{\partial x'^2}, \frac{\Box', H_1'}{\phi(x)} \right\}, \]  

(2.20)

where \( H_1' = i t^2 \partial_x \) is the generator of the pure Galilei transformations. The weight function \( \Phi(x) \) is (see [6, 7, 8])

\[ \Phi_{G_1}(x) = \frac{\phi(x' - x)}{\phi(x)} = \exp \left\{ -\frac{i}{\gamma} \left[ (1 - \gamma) k \cdot x - \beta \omega \left( n_x t - \frac{x}{c} \right) \right] \right\}, \]  

(2.21)

where \( k = (k, k_4), k = \omega n / c \) is the wave vector, \( n \) is the wave front guiding vector, \( \omega \) is the wave frequency, \( k_4 = i \omega / c, k_1' = (k_1 + i \beta k_4) / \gamma, k_2' = k_2 / \gamma, k_3' = k_3 / \gamma, k_4' = k_4, k'^2 = k^2 - \text{inv} \), where \( \text{inv} \) means invariant. (For comparison, in the relativistic case we have \( k_1' = (k_1 + i \beta k_4) / (1 - \beta^2)^{1/2}, k_2' = k_2, k_3' = k_3, k_4' = (k_4 - i \beta k_1) / (1 - \beta^2)^{1/2}, k'^2 = k^2 + k_4'^2 = k^2 + k_4'^2 \) as is well known.)

### 3. Comparison of the results

Table 3.1 illustrates the results obtained above.

<table>
<thead>
<tr>
<th>Group</th>
<th>( P_{10} )</th>
<th>( W_{11} )</th>
<th>( I )</th>
<th>( C_4 )</th>
<th>( G_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W \Phi(x) )</td>
<td>1</td>
<td>( \rho^l )</td>
<td>( x^2 )</td>
<td>( \sigma(x) )</td>
<td>( \exp{-i[(1 - \gamma) k \cdot x - \beta \omega(n_x t - x/c)] / \gamma } )</td>
</tr>
</tbody>
</table>

For the different transformations \( x' = x'(x) \), the weight functions \( \Phi(x) \) may be found in a similar way.

Note that in the symmetry theory of the D’Alembert equation, conditions (2.6) for transforming this equation into itself combine the requirements formulated by various authors, as can be seen in Table 3.2, where \( m_{\alpha}, m_0 \) are some numbers, \( D_{\alpha\beta} \) and \( M_{\alpha\beta} \) are the \( 6 \times 6 \) numerical matrices.

According to Table 3.2 for the field \( \phi' = \phi \) with conformal dimension \( l = 0 \) and the linear homogeneous coordinate transformations from the group \( L_x \Delta \in W_{11} \) with \( \rho = (1 - \beta^2)^{1/2} \), the formulae were proposed by Voigt [13] and cited by Pauli [10]. In the plain waves case, they correspond to the transformations of the 4-vector \( k = (k, k_4) \) and proper frequency \( \omega_0 \) according to the law \( k_1' = (k_1 + i \beta k_4) / \rho (1 - \beta^2)^{1/2}, k_2' = k_2 / \rho, k_3' = k_3 / \rho, k_4' = (k_4 - i \beta k_1) / \rho (1 - \beta^2)^{1/2}, \omega_0' = \omega_0 / \rho, k' x' = k x \)-inv. In the case of the \( W_{11} \)-coordinate transformations belonging to the set of arbitrary transformations \( x' = x'(x) \), the requirements for the one component field with \( l = 0 \) were found by Umov [12]. The requirement that the second derivative \( \partial^2 \phi'/\partial x'^2, \partial^2 \phi'/\partial \phi_{\alpha} \partial \phi_{\beta} = 0 \) with \( \Phi = 1 \) is turned into zero was introduced by Di Jorio [3]. The weight function \( \Phi \neq 1 \) and the set (2.6) were proposed by the author of the present work [6, 7, 8].

By now only the D’Alembert equation symmetries corresponding to the linear systems of the type (2.8), (2.12), and (2.15) have been well studied. These are the well-known relativistic and conformal symmetry of the equation. The investigations corresponding to the linear conditions (2.6) are much more scanty and presented only
TABLE 3.2

<table>
<thead>
<tr>
<th>Author</th>
<th>Coordinates transform.</th>
<th>Group</th>
<th>Conditions of invariance</th>
<th>Fields transform.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voigt [13]</td>
<td>(x_j' = A_{jk} x_k)</td>
<td>(L_6 X \Delta_1)</td>
<td>(A'<em>{ji} L</em>{k1} = \rho'^2 \delta_{jk})</td>
<td>(\phi' = \phi)</td>
</tr>
<tr>
<td>Umov [12]</td>
<td>(x_j' = x_{j'}(x))</td>
<td>(W_{11})</td>
<td>(\frac{\partial x_j'}{\partial x_{j'}} \frac{\partial x_k}{\partial x_{k'}} = \rho'^2 \delta_{jk})</td>
<td>(\phi' = \phi)</td>
</tr>
<tr>
<td>Di Jorio [3]</td>
<td>(x_j' = L_{jk} x_k + a_j)</td>
<td>(P_{10})</td>
<td>(L'<em>{ji} L'</em>{ki} = \delta_{jk})</td>
<td>(\phi' = m_\alpha \phi_\alpha + m_0;)</td>
</tr>
<tr>
<td>Kotel’nikov [6, 7, 8]</td>
<td>(x_j' = x_{j'}(x))</td>
<td>(C_4)</td>
<td>(\frac{\partial^2 \phi'<em>\alpha}{\partial \xi</em>{\beta} \partial \xi_{\gamma}} = 0)</td>
<td>(\xi_{\alpha} = \psi \phi_{\alpha})</td>
</tr>
<tr>
<td></td>
<td>(x_j' = x_{j'}(x))</td>
<td>(G_1)</td>
<td>(\frac{\partial^2 \phi'<em>\alpha}{\partial \xi</em>{\beta} \partial \xi_{\gamma}} = 0)</td>
<td>(\xi_{\alpha} = \psi \phi_{\alpha})</td>
</tr>
</tbody>
</table>

in [6, 7, 8]. The publications corresponding to the nonlinear conditions (2.3) are completely absent. The difficulties arising here are connected with the analysis of compatibility of the set (2.3) containing the nonlinear partial differential equation.

4. Conclusion. It is shown that with the generalized understanding of the symmetry according to Definition 2.1, the D’Alembert equation for one component field is invariant with respect to any arbitrary reversible coordinate transformations \(x' = x'(x)\). In particular, they contain the transformations of the conformal and Galilei groups realizing the type of standard and generalized symmetry for \(\Phi(x) = \phi'(x' \to x)/\phi(x)\).

The concept of partial differential equations symmetry is conventional.

References


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