LOCAL COMPLETENESS OF $\ell_p(E)$, $1 \leq p < \infty$

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We study the heredity of local completeness and the strict Mackey convergence property from the locally convex space $E$ to the space of absolutely $p$-summable sequences on $E$, $\ell_p(E)$ for $1 \leq p < \infty$.

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1. Introduction. In 1956, Grothendieck [5], introduced the Banach-valued sequence space $\ell_p(E)$, the space of absolutely $p$-summable sequences on a Banach space $E$, where he discussed tensor products of $\ell_p$ and $E$, with $1 \leq p \leq \infty$. Later, in 1969 Pietsch [8] used Banach-valued sequence spaces $\ell_p(E)$, to study $p$-summing operators between Banach spaces, also see Diestel et al. [2]. In this paper, we discuss how local completeness and the strict Mackey convergence condition of $E$ imply local completeness and the strict Mackey convergence condition in $\ell_p(E)$ in the case $1 \leq p < \infty$. The case $p = \infty$ was studied in [1].

2. Definitions and notation. Throughout this paper, $(E,t)$ denotes a Hausdorff locally convex space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{\rho_j\}_{j \in J}$ denotes the family of continuous seminorms associated with the topology $t$ on $E$.

Let $D \subset E$ be a bounded, closed, and absolutely convex set. Denote by $E_D = \cup_{k=1}^n kD$, and for each $x \in E_D$, $\rho_D(x) = \inf \{r > 0 : x \in rD\}$, the Minkowski seminorm associated with $D$. Now $E_D \subset E$ and the boundedness of $D$ implies that $i : (E_D,\rho_D) \to (E,t)$ is continuous, and $\rho_D$ is a norm so that, for every $j \in J$ there exists $r_j \in \mathbb{R}^+$ such that $\rho_j|_{E_D} \leq r_j \rho_D$.

**Remark 2.1.** For each $D \subset E$ bounded, closed, and absolutely convex, the family of seminorms $\{\rho_j\}_{j \in J}$ can be replaced by an equivalent family $\{\rho'_j\}_{j \in J}$ such that $\rho'_j \leq \rho_D$. To construct the family $\{\rho'_j\}_{j \in J}$ we know that there exists $r_j > 0$ such that $\rho_j(x) \leq r_j \rho_D(x)$ for every $x \in E_D$ so it suffices to take $\rho'_j = (1/r_j)\rho_j$ if $r_j > 1$, and we will have $\rho'_j \leq \rho_D$, for every $j \in J$. For simplicity we will always work with an equivalent family of seminorms, also denoted by $\{\rho_j\}_{j \in J}$ such that $\rho_j(x) \leq \rho_D(x)$ holds for every $j \in J$ and $x \in E_D$.

A bounded, closed, and absolutely convex set $D \subset E$, called a disk, is a Banach disk if $(E_D,\rho_D)$ is a Banach space. If every bounded set $A \subset E$ is contained in a Banach disk we say that $E$ is locally complete. Let $(E,t)$ satisfies the strict Mackey convergence condition if for every bounded set $A \subset E$, there exists a disk $D$ that contains $A$ such that the topologies of $(E,t)$ and $(E_D,\rho_D)$ agree on $A$. 
Every metrizable space satisfies the strict Mackey convergence condition, [7]. In addition, the strict Mackey convergence condition is preserved under the formation of closed subspaces, countable products, and countable direct sums, [6]. The strict Mackey convergence condition for webbed spaces is studied in [3, 4].

**Remark 2.2.** Using the family of seminorms \( \{ \rho_j \}_{j \in J} \) it is easy to see that the strict Mackey convergence condition is equivalent to: for each \( \mathcal{D} \) there exists \( j_0 \in J \) such that \( \rho_{j_0|\mathcal{D}} = \rho_{\mathcal{D}} \).

Let \( p \) be a real number such that \( 1 \leq p < \infty \). The space \( \ell_p(\mathcal{E}) \) of absolutely \( p \)-summable sequences on \( \mathcal{E} \) is

\[
\ell_p(\mathcal{E}) = \left\{ (x_n) \in \mathcal{E} : \sum_{n=1}^{\infty} |x_n|^p < \infty, \forall j \in J \right\}.
\]  

The family of seminorms \( \rho_{\rho_j}((x_n)) = \sum_{n=1}^{\infty} |x_n|^p \) induces a topology of locally convex space in \( \ell_p(\mathcal{E}) \); we will denote by \( \tau \) this topology.

The space \( \ell_p(\mathcal{E}_D) \) is defined by \( \ell_p(\mathcal{E}_D) = \{(x_n) \in \mathcal{E}_D : \sum_{n=1}^{\infty} |x_n|^p < \infty \} \) and endowed with the topology generated by the norm

\[
\rho_{\mathcal{E}_D}((x_n)) = \left[ \sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}.
\]  

We denote \( A_D = \{(x_n) \in \ell_p(\mathcal{E}) : (x_n)_{n \in \mathbb{N}} \subset \mathcal{D} \} \).

Note that \( \rho_{\rho_j}|_{\ell_p(\mathcal{E}_D)} \leq \rho_{\mathcal{E}_D} \) for every \( j \in J \) since \( \rho_j|_{\mathcal{E}_D} \leq \rho_{\mathcal{E}_D} \).

### 3. Bounded sets

In this section, we characterize the bounded sets of \( \ell_p(\mathcal{E}) \) in terms of the bounded sets of \( \ell_p(\mathcal{E}) \).

**Lemma 3.1.** Let \( \mathcal{D} \) be a disk in \((E, t)_\mathcal{D}\); then

(i) \( \ell_p(\mathcal{E}_D) \subseteq \{(x_n) \in \ell_p(\mathcal{E}) : (x_n) \subset k\mathcal{D} \text{ for some } k \in \mathbb{N} \} \);

(ii) if there exists \( j_0 \in J \), depending on \( \mathcal{D} \), such that \( \rho_{j_0|\mathcal{D}} = \rho_{\mathcal{D}} \) (i.e., the strict Mackey convergence condition holds), then \( \{(x_n) \in \ell_p(\mathcal{E}) : (x_n) \subset k\mathcal{D} \text{ for some } k \in \mathbb{N} \} \subset \ell_p(\mathcal{E}_D) \).

**Proof.**  
(i) Let \( (x_n) \in \ell_p(\mathcal{E}_D) \). Then \( \sum_{n=1}^{\infty} (\rho_{\mathcal{D}}(x_n))^p < \infty \) so that given \( \varepsilon = 1 \) there exists \( n_0 \in \mathbb{N} \), such that for each \( n > n_0 \), we have \( \rho_{\mathcal{D}}(x_n) \leq (\sum_{n=1}^{\infty} \rho_{\mathcal{D}}^p(x_n))^{1/p} \leq 1 \) which means that \( x_n \in \mathcal{D} \) for every \( n > n_0 \).

Now for \( i = 1, 2, \ldots, n_0 \) there exists \( k_i \geq 1 \) such that \( x_i \in k_i\mathcal{D} \). We take \( k = \max\{1, k_1, \ldots, k_{n_0}\} \). Then \( (x_n) \subset k\mathcal{D} \) and we have \( \ell_p(\mathcal{E}_D) \subset \{(x_n) \in \ell_p(\mathcal{E}) : (x_n) \subset k\mathcal{D} \text{ for some } k \in \mathbb{N} \} \).

(ii) Let \( (x_n) \in \{(y_n) \in \ell_p(\mathcal{E}) : (y_n) \subset k\mathcal{D} \text{ for some } k \in \mathbb{N} \} \). Thus \( x_n \in \mathcal{E}_D \) for every \( n \in \mathbb{N} \) since \( (x_n) \subset k\mathcal{D} \).

Now observe that \( \sum_{n=1}^{\infty} \rho_{\mathcal{D}}^p(x_n) = \sum_{n=1}^{\infty} \rho_{j_0|\mathcal{D}}^p(x_n) < \infty \) since \( (x_n) \in \ell_p(\mathcal{E}) \). Hence in this case we have the equality \( \ell_p(\mathcal{E}_D) = \{(x_n) \in \ell_p(\mathcal{E}) : (x_n) \subset k\mathcal{D} \text{ for some } k \in \mathbb{N} \} \).

**Remark 3.2.** Note that \( kA_D = A_{k\mathcal{D}} \) for every \( k \in \mathbb{N} \).
COROLLARY 3.3. If $E$ satisfies the strict Mackey convergence condition, then $\ell_p(E)_{AD} = \ell_p(E_D)$.

PROOF. It follows from the equality in the proof of Lemma 3.1(ii) that $\ell_p(E)_{AD} \subset \ell_p(E_D)$. Now let $(x_n) \in \ell_p(E_D)$. Then by Lemma 3.1(i), $(x_n) \in kD$ for some $k \in \mathbb{N}$ so $\{x_n\} \subset AkD = kAD$ and $(x_n) \in \ell_p(E)_{AD}$.

REMARK 3.4. If $(E,t)$ satisfies the strict Mackey convergence condition, then

$$\ell_p(E)_{AD} = \ell_p(E) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset A_{kD} \text{ for some } k \in \mathbb{N}\}.$$  (3.1)

LEMMA 3.5. (i) $\rho_{AD}((x_n)) = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$;
(ii) $\rho_{AD}((x_n)) \leq \rho_D((x_n))$ for every $(x_n) \in \ell_p(E_D)$.

PROOF. (i) Let $s = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$. Then $\{x_n\} \subset sD$ so $\{x_n\} \subset A_{sD} = sAD$ and then $\rho_D((x_n)) \leq s$. Now take $r = \rho_{AD}((x_n))$. Then $\{x_n\} \subset rA_D = rAD$ and then $\rho_D((x_n)) = \rho_{AD}((x_n))$.
(ii) $\rho_{AD}((x_n)) = (\sum_{n=1}^{\infty} D_p(x_n))^{1/p} \geq \rho_D(x_n)$ for every $n \in \mathbb{N}$. Using (i) we have $\rho_{AD}((x_n)) = \rho_{AD}((x_n))$.

Note that $A_D$ is not bounded in $\ell_p(E)$; we need to construct a “smaller” set, in the sense of boundedness.

Define for each $j \in J$ and $m \in \mathbb{N}$ the set $A_D(j,m) = \{(x_n) \in A_D : \rho_{AD}((x_n)) \leq m\}$ and for each $B \subset \ell_p(E)$, let $B^* = \{x \in E : x \in (x_n)\}$ and $(x_n) \in B$.

The next proposition gives a way to look at the bounded sets in $\ell_p(E)$.

PROPOSITION 3.6. If $\beta = \{D_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of bounded disks in $E$, then $\{C = \cap_{j \in J} A_D((j,m_j)) : \lambda \in \Lambda, (m_j) \in \mathbb{N}^J\}$ is a fundamental system of $\tau$-bounded sets in $\ell_p(E)$.

PROOF. Let $B \subset \ell_p(E)$ be a bounded set. Then $B^*$ is bounded in $E$ so $B^* \subset D_\lambda$ for some $\lambda$. For each $x \in B^*$, if $x \in (x_n)$ then there is some $s_j$ such that $\rho_j(x) \leq \rho_{AD}((x_n)) \leq s_j$ so that $\rho_j(B) \leq s_j$. Now let $m_j \in \mathbb{N}$ be such that $s_j \leq m_j$. We have $B \subset C = \cap_{j \in J} A_D((j,m_j))$.

REMARK 3.7. (i) If $D$ is bounded in $E$, then for each $j \in J$, by Remark 2.1 $\rho_{j,D}^\beta \leq \rho_D$.
(ii) If $C$ is bounded in $\ell_p(E)$, then for each $j \in J$, by Remark 2.1 $\rho_{j,D}^\beta \leq \rho_C$.

4. Main results

PROPOSITION 4.1. If for some $D$ there exists $j_0 \in J$, such that $\rho_{j_0,D} = \rho_D$ in $E$, then $\rho_{\rho_{j_0},C} = \rho_C$ where $C = \cap_{j \in J} A_D((j,m_j))$ in $\ell_p(E)$. Equivalently, if $E$ satisfies the strict Mackey convergence condition, then $\ell_p(E)$ also satisfies the strict Mackey convergence condition.

PROOF. Let $(x_n) \in C$. Then $s = \rho_{j_0,D}((x_n)) = (\sum_{n=1}^{\infty} D_p((x_n)))^{1/p} = (\sum_{n=1}^{\infty} D_p((x_n)))^{1/p} \geq \rho_D(x_n) \geq \rho_{j_0,D}(x_n)$ for every $j \in J$ and $n \in \mathbb{N}$. So we have $(x_n) \in \cap_{j \in J} A_D((j,s)) = s[\cap_{j \in J} A_D((j,1))] \subset sC$. Thus $\rho_C((x_n)) \leq s = \rho_{j_0,D}(x_n)$ and since $C$ is bounded in $\ell_p(E)$ we have $\rho_{j_0,D} \leq \rho_C$ for each $j \in J$; now $\rho_{j,D} \leq \rho_C$ for every $j \in J$, so for $j_0$ we have $\rho_{j_0,D} = \rho_C$. 

LOCAL COMPLETENESS OF $\ell_p(E)$, 1 ≤ $p < \infty$ 653
Notice that if $B$ is a bounded set in $\ell_p(E)$, then $\rho_{\rho_j}(B) \leq m_j$ for all $j \in J$ with $m_j \in \mathbb{N}$ and then $B \subset \cap_{j \in J} A_B^*(j, m_j)$.

This gives the property we need to characterize the bounded sets in $\ell_p(E)$. □

**THEOREM 4.2.** If $E$ is locally complete and satisfies the strict Mackey convergence condition, then $(\ell_p(E)_C, \rho_C)$ where $C = \cap_{j \in J} A_D(j, m_j)$ in $\ell_p(E)$, is a Banach space so $\ell_p(E)$ is locally complete.

**PROOF.** Let $D$ be a bounded closed disk such that $(E_D, \rho_D)$ is a Banach space and let $C = \cap_{j \in J} A_D(j, m_j)$. By Remark 2.1 there is a $j_0 \in J$ such that $\rho_{j_0}|_D = \rho_D$. We will show that $(\ell_p(E)_C, \rho_C)$ is a Banach space. By Corollary 3.3 we have $\ell_p(E)_D = \ell_p(E_D)$ and since $C \subset A_D$, $\ell_p(E)_C \subset \ell_p(E)_D$. Let $(x_n^k)_{k \in \mathbb{N}} \subset \ell_p(E)_C$ be a $\rho_C$-Cauchy sequence. Thus for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$ we have $\rho_C((x_n^k) - (x_m^k)) < \varepsilon$. Using Remark 3.7(ii) we have that $\rho_{\rho_j}|_D = \rho_C$. Hence $(x_n^k)$ is also a $\rho_{\rho_j}$-Cauchy sequence and then a $\rho_{\rho_{j_0}}$-Cauchy sequence. Thus $\rho_{D}(x_n^k - x_m^k) = \rho_{j_0}(x_n^k - x_m^k) \leq \rho_{j_0}(\lim_{n \to \infty} x_n^k - \lim_{m \to \infty} x_m^k)$, then the sequence $(x_n^k)_{k \in \mathbb{N}}$ for every $n \in \mathbb{N}$ is also a $\rho_{D}$-Cauchy sequence in $(E_D, \rho_D)$ which is a Banach space, so there exists $z^k$ in $E_D$ such that $(x_n^k)$ converges to $z^k$ with respect to the norm $\rho_D$. Using Remark 3.7(i) we have $\rho_{\rho_{j_0}} \leq \rho_{j_0}$. Hence, we have the following claims.

**CLAIM 1.** We have that $(x_n^k)$ converges to $z^k$ with respect to the seminorm $\rho_j$ for every $j \in J$.

**CLAIM 2.** Consider the sequence formed by the $(z^k)_{k \in \mathbb{N}} \in \ell_p(E_D)$. We compute

$$
\sum_{k=1}^{\infty} (\rho_D(z^k))^p = \lim_{m \to \infty} \sum_{k=1}^{m} (\rho_D(z^k))^p
= \lim_{m \to \infty} \rho_{j_0}(z^k)^p
= \lim_{m \to \infty} \sum_{k=1}^{m} \rho_{j_0}(\lim_{n \to \infty} x_n^k)^p
= \lim_{m \to \infty} \lim_{n \to \infty} \sum_{k=1}^{m} \rho_{j_0}(x_n^k)^p
\leq \lim_{m \to \infty} \lim_{n \to \infty} \sum_{k=1}^{m} \rho_{j_0}(x_n^k)^p
= \lim_{n \to \infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p
\leq \lim_{n \to \infty} \rho_{\rho_{j_0}}((x_n))
\leq \varepsilon + \rho_{\rho_{j_0}}((x_N)) < \infty, \text{ for some } N \in \mathbb{N}.
$$

(4.1)

In this last inequality we used $x_n = (x_n^k)_{k \in \mathbb{N}}$ and since it is a $\rho_{\rho_{j_0}}$-Cauchy sequence, given $\varepsilon > 0$, $\rho_{\rho_{j_0}}(x_n^k - x_m^k) \leq \rho_{\rho_{j_0}}(x_n^k - (x_m^k)) < \varepsilon$ for every $n, m > N$, so $\rho_{\rho_{j_0}}((x_n)) \leq \varepsilon + \rho_{\rho_{j_0}}((x_N))$. Notice that $(x_n)$ is a $\rho_{\rho_j}$-Cauchy sequence for every $j \in J$. 


Therefore for \( j_0 \) and consequently for \( \rho_{pD} \), then for every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( \rho_D(x_{n}^k - z^k) = \rho_D(x_{n}^k - \lim_{m \to \infty} x_{m}^k) = \lim_{m \to \infty} \rho_D(x_{n}^k - x_{m}^k) < \varepsilon \).

**Claim 3.** The sequence \((x_{n}^k)\) converges to \((z^k)_{k \in \mathbb{N}}\) in \( \ell_p(E_D) \). Since

\[
\rho_{pD}(x_{n}^k - (z^k)_{k}) = \left[ \sum_{k=1}^{\infty} \rho_D^p(x_{n}^k - z^k) \right]^{1/p}
\leq \left[ \sum_{k=1}^{N} \rho_D^p(x_{n}^k - z^k) + \frac{\varepsilon^p}{2} \right]^{1/p}
\leq \left( \frac{\varepsilon^p + \cdots + \varepsilon^p}{2N} \right)^{1/p}
\leq \varepsilon,
\]

\((4.2)\)

In the first inequality we used Claim 2. This completes the proof of the convergence.

**Claim 4.** We have \((z^k)_{k \in \mathbb{N}} \in \ell_p(E)_{C} \). \((x_{n}^k)_{k \in \mathbb{N}} \) is a \( p \)-Cauchy sequence so it is bounded and there is an \( s \in \mathbb{N} \) such that \((x_{n}^k) \subseteq sC \). Using Claim 3, \((x_{n}^k)\) converges to \((z^k)\) in \( \ell_p(E)_{C} \) with respect to \( \rho_{pD} \) and since \( \rho_{p_j}|_{\ell_p(E)} \leq \rho_{pD} \) for every \( j \in J \) the sequence \((x_{n}^k)\) is \( \tau \)-convergent to \((z^k)\), it is convergent for each \( \rho_{p_j} \). Now for each \( \varepsilon > 0 \) there exists \( N_j \) such that \( \rho_{p_j}((z^k)) \leq \rho_{p_j}((z^k) - (x_{n}^k)) + \rho_{p_j}((x_{n}^k)) < \varepsilon + s \) for every \( j \in J \) and \( n \geq N_j \), this means that \((z^k) \in sC \subseteq \ell_p(E)_{C} \).

**Claim 5.** The sequence \((x_{n}^k)\) converges to \((z^k)_{k \in \mathbb{N}}\) in \( \ell_p(E) \). Let \( \varepsilon > 0 \), since \((x_{n}^k)\) is a \( p \)-Cauchy sequence, there is \( N \in \mathbb{N} \) such that \((x_{n}^k) - (x_{m}^k) \in \varepsilon C \) for every \( n, m \geq N \). \( C \) is \( \tau \)-closed so \((x_{n}^k) - (\tau - \lim(x_{n}^k)) \in \varepsilon C \); then \((x_{n}^k) - (z^k) \in \varepsilon C \) for every \( n \geq N \) which means \( \rho_{C}((x_{n}^k) - (z^k)) \leq \varepsilon \) for every \( n \geq N \).

Notice that this is true for every \( 1 \leq p < \infty \). The case \( p = \infty \) also follows from this and we get the characterization given in [1], although under a stronger hypothesis. Here we need \( E \) to satisfy the strict Mackey convergence condition. \( \square \)

**Lemma 4.3.** If \( D \subseteq E \) is \( t \)-complete and the net \( \{x_{\lambda}\}_{\Lambda} \) is a \( \tau \)-Cauchy net bounded with respect to \( \rho_{C} \), that is if there exists \( s \in \mathbb{N} \) such that \( \{x_{\lambda}\}_{\Lambda} \subseteq sC \) then there exists \( z \in 2sC \) such that \( x_{\lambda} \) converges to \( z \) with respect to the \( \tau \) topology in \( \ell_p(E) \).

**Proof.** Let \( \{x_{\lambda}\}_{\Lambda} \) be a \( \tau \)-Cauchy net, \( x_{\lambda} = (x_{\lambda}^1, x_{\lambda}^2, \ldots) \), then for every \( \varepsilon > 0 \) there exists \( \lambda_j \in \Lambda \) such that for every \( j \in J \), \( \rho_{j}(x_{\lambda}^k - x_{\lambda'}^k) \leq \rho_{p_j}(x_{\lambda} - x_{\lambda'}) < \varepsilon \) for every \( \lambda, \lambda' \geq \lambda_j \) and \( k \in \mathbb{N} \). So \( \{x_{\lambda}^k\}_{\Lambda} \subseteq D \) is \( t \)-Cauchy for each \( k \in \mathbb{N} \), and since \( D \) is complete there is a \( z^k \) such that \( x_{\lambda}^k \) converges to \( z^k \) with respect to the topology \( t \) for each \( k \in \mathbb{N} \). Let \( z = (z^1, z^2, \ldots) \). Then \( z \in D \), and for each \( j \in J \) and \( k \in \mathbb{N} \) we have \( \rho_{j}(x_{\lambda}^k - z^k) = \rho_{j}(x_{\lambda}^k - (\rho_{j}(\lim_{\lambda'} x_{\lambda'}^k))) = \lim_{\lambda'} \rho_{j}(x_{\lambda}^k - x_{\lambda'}^k) \), so raising to the \( p \)th power and adding with respect to \( k \) we have

\[
\sum_{k=1}^{\infty} \rho_{j}(x_{\lambda}^k - z^k)^p = \lim_{n \to \infty} \sum_{k=1}^{n} \rho_{j}(x_{\lambda}^k - z^k)^p
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \lim_{\lambda'} \rho_{j}(x_{\lambda}^k - x_{\lambda'}^k)^p
\]
= \lim_{n \to \infty} \sum_{k=1}^{n} \rho_j(x^k_\lambda - z^k)^p \\
\leq \lim_{n \to \infty} \sum_{k=1}^{n} \rho_j(x^k_\lambda - z^k)^p \\
= \lim_{n \to \infty} \rho_j(x_\lambda - x_N) < \varepsilon^p, \\
(4.3)

for every \( \lambda \geq \lambda_j \).

So we have \( \rho_j(x_\lambda - z)^p = \sum_{k=1}^{\infty} \rho_j(x^k_\lambda - z^k)^p < \varepsilon^p \), for every \( \lambda \geq \lambda_j \). This means that \( x_\lambda \) converges to \( z \) with respect to the topology \( \tau \). We still need to prove that \( z \in \ell_p(E) \)

\[
\rho_j(z)^p = \sum_{k=1}^{\infty} \rho_j(z^k)^p \\
= \sum_{k=1}^{\infty} \rho_j(z^k + x^k_\lambda - x^k_\lambda)^p \\
\leq \sum_{k=1}^{\infty} 2^p \left[ \rho_j(z^k - x^k_\lambda)^p + \rho_j(x^k_\lambda)^p \right] \\
= 2^p \sum_{k=1}^{\infty} \rho_j(z^k - x^k_\lambda)^p + 2^p \sum_{k=1}^{\infty} \rho_j(x^k_\lambda)^p \\
< 2^p \varepsilon^p + 2^p \rho_j(x_\lambda)^p \\
\leq 2^p \varepsilon^p + 2^p m_j
\]

\((x_\lambda \in C = \cap_{j \in J} A_D(j,m_j))\), then if we let \( \varepsilon \to 0 \) we get \( \rho_j(z) \leq 2m_j \), and finally \( z \in 2C \subset \ell_p(E) \).

**Theorem 4.4.** If \( D \) is \( t \)-complete, then \( \ell_p(E)_C \) is \( \rho_C \)-complete.

**Proof.** Let \((x^k_n)\) be a \( \rho_C \)-Cauchy sequence; it is clearly \( \rho_C \)-bounded and \( \tau \)-Cauchy, so \((x^k_n) \subset sC\) for some \( s \in \mathbb{N} \). Then by **Lemma 4.3**, there is a \( z = (z^k) \in 2sC \subset \ell_p(E)_C \) such that the sequence \((x^k_n)\) converges to \( z \) with respect to the topology \( \tau \). Note that \( A_D \) is \( \tau \)-closed so \( A_D(j,m) \) is also \( \tau \)-closed for every \( j \in J \) and \( m \in \mathbb{N} \); then \( C = \cap_{j \in J} A_D(j,m_j) \) is \( \tau \)-closed. For \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) such that \((x^k_n) - (x^k_m) \in \varepsilon C\) for every \( n, m \geq N \), and since \( C \) is \( \tau \)-closed \((x^k_n) - (\tau - \lim(x^k_m)) \in \varepsilon C \) then \((x^k_n) - (z^k) \in \varepsilon C\) for every \( n \geq N \). This means that \((x^k_n)\) converges to \((z^k)\) with respect to \( \rho_C \).

**Theorem 4.5.** If \( E \) is \( t \)-complete, then \( \ell_p(E) \) is \( \tau \)-complete.

**Proof.** The proof of **Lemma 4.3** can be repeated here to get the \( \tau \)-completeness of \( \ell_p(E) \).

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LOCAL COMPLETENESS OF $\ell_p(E)$, $1 \leq p < \infty$

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