FUZZY $c\gamma$-OPEN SETS AND FUZZY $c\gamma$-CONTINUITY IN FUZZIFYING TOPOLOGY

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The concepts of fuzzy $c\gamma$-open sets and fuzzy $c\gamma$-continuity are introduced and studied in fuzzifying topology and by making use of these concepts, some decompositions of fuzzy continuity are introduced.

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1. Introduction. In [8, 9, 10], Ying introduced the concept of fuzzifying topology with the semantic method of continuous valued logic. All the conventions in [8, 9, 10] are good in this paper. Andrijević [3] introduced the concepts of $b$-open sets in general topology. We note that the concepts of $\gamma$-open sets and $\gamma$-continuity are considered by Hanafy [4] to fuzzy topology. In [7], the concepts of fuzzy $\gamma$-open sets and fuzzy $\gamma$-continuity are introduced and studied in fuzzifying topology. In the present paper, we define and study the concepts of $c\gamma$-open sets and $c\gamma$-continuity in fuzzifying topology. The main purpose of the present paper is to obtain decompositions of fuzzy continuity in fuzzifying topology by making use of fuzzy $\gamma$-continuity and fuzzy $c\gamma$-continuity.

2. Preliminaries. We present the fuzzy logical and corresponding set theoretical notations due to Ying [8, 9].

For any formulae $\varphi$, the symbol $[\varphi]$ means the truth value of $\varphi$, where the set of truth values is the unit interval $[0, 1]$. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation.

The original formulae of fuzzy logical and corresponding set theoretical notations are

(1) (a) $[\alpha] = \alpha (\alpha \in [0, 1])$;
(b) $[\varphi \land \psi] := \min ([\varphi], [\psi])$;
(c) $[\varphi \rightarrow \psi] := \min (1, 1 - [\varphi] + [\psi])$.

(2) If $A \in \mathcal{F}(X)$, $[x \in \tilde{A}] := \tilde{A}(x)$.

(3) If $X$ is the universe of discourse, $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition the following derived formulae are given:

(1) $[\neg \varphi] := [\varphi \rightarrow 0] = 1 - [\varphi]$;
(2) $[\varphi \lor \psi] := [\neg (\neg \varphi \land \neg \psi)] := \max ([\varphi], [\psi])$;
(3) $[\varphi \land \psi] := [(\varphi \land \psi) \land (\psi \land \varphi)]$;
(4) $[\varphi \land \psi] := [\neg (\neg \varphi \land \neg \psi)] := \max (0, [\varphi] + [\psi] - 1)$;
(5) $[\varphi \lor \psi] := [\neg \varphi \lor \psi] = [\neg (\neg \varphi \land \neg \psi)] = \min (1, [\varphi] + [\psi])$;
(6) $[\exists x \varphi(x)] := [\neg \forall x \neg \varphi(x)] = \sup_{x \in X} [\varphi(x)]$;
(7) if $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$, then
(a) \([\tilde{A} \subseteq \tilde{B}] := [\forall x (x \in \tilde{A} - x \in \tilde{B})] = \inf_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x));\)
(b) \([\tilde{A} \equiv \tilde{B}] := [\tilde{A} \subseteq \tilde{B}] \land (\tilde{B} \subseteq \tilde{A});\)
(c) \([\tilde{A} \equiv \tilde{B}] := [\tilde{A} \subseteq \tilde{B}] \land (\tilde{B} \subseteq \tilde{A}),\)

where \(\mathcal{F}(X)\) is the family of all fuzzy sets in \(X\).

We do not often distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. We now give the following definitions and results in fuzzifying topology which are used in the sequel.

**Definition 2.1** (see [8]). Let \(X\) be a universe of discourse, \(P(X)\) the family of subsets of \(X\), and \(\tau \in \mathcal{F}(P(X))\) satisfy the following conditions:

1. \(\tau(X) = 1, \tau(\emptyset) = 1;\)
2. for any \(A, B, \tau(A \cap B) \geq \tau(A) \land \tau(B);\)
3. for any \([A_\lambda : \lambda \in \Lambda], \tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda).\)

Then \(\tau\) is called a fuzzifying topology and \((X, \tau)\) is a fuzzifying topological space.

**Definition 2.2** (see [8]). The family of fuzzifying closed sets, denoted by \(F \in \mathcal{F}(P(X))\), is defined as \(A \in F := X \sim A \in \tau\), where \(X \sim A\) is the complement of \(A\).

**Definition 2.3** (see [8]). Let \(x \in X\). The neighborhood system of \(x\), denoted by \(N_x \in \mathcal{F}(P(X))\), is defined as \(N_x(A) = \sup_{x \in B \subseteq A} \tau(B)\).

**Definition 2.4** (see [8, Lemma 5.2]). The closure \(\tilde{A}\) of \(A\) is defined as \(\tilde{A}(x) = 1 - N_x(X \sim A)\).

In [8, Theorem 5.3], Ying proved that the closure \(\tilde{\cdot} : P(X) \to \mathcal{F}(X)\) is a fuzzifying closure operator (see [8, Definition 5.3]) since its extension \(\tilde{\cdot} : \mathcal{F}(X) \to \mathcal{F}(X), \tilde{A} = \cup_{\alpha \in [0, 1]} \alpha \tilde{A}_\alpha, \tilde{A} \in \mathcal{F}(X)\) satisfies the following Kuratowski closure axioms:

1. \(\vdash \emptyset \equiv \emptyset;\)
2. for any \(\tilde{A} \in \mathcal{F}(X), \vdash \tilde{A} \subseteq \tilde{A};\)
3. for any \(\tilde{A}, \tilde{B} \in \mathcal{F}(X), \vdash \tilde{A} \cup \tilde{B} \equiv \tilde{A} \cup \tilde{B};\)
4. for any \(\tilde{A} \in \mathcal{F}(X), \vdash (\tilde{A}) \subseteq \tilde{A},\)

where \(\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}\) is the \(\alpha\)-cut of \(A\) and \(\alpha \tilde{A}(x) = \alpha \land \tilde{A}(x)\).

**Definition 2.5** (see [9]). For any \(A \in P(X)\), the interior of \(A\), denoted by \(A^* \in \mathcal{F}(P(X))\), is defined as follows: \(A^*(x) = N_x(A)\).

From [8, Lemma 3.1] and the definitions of \(N_x(A)\) and \(A^*\) for \(A \in P(X)\) we have \(\tau(A) = \inf_{x \in A} A^*(x)\).

**Definition 2.6** (see [5]). For any \(\tilde{A} \in \mathcal{F}(X), \vdash (\tilde{A})^* \equiv X \sim (X \sim \tilde{A}).\)

**Lemma 2.7** (see [5]). If \([\tilde{A} \subseteq \tilde{B}] = 1, then

1. \(\vdash \tilde{A} \subseteq \tilde{B};\)
2. \(\vdash (\tilde{A})^* \subseteq \tilde{B}\).

**Lemma 2.8** (see [5]). Let \((X, \tau)\) be a fuzzifying topological space. For any \(\tilde{A}, \tilde{B},\)

1. \(\vdash X^* = X;\)
2. \(\vdash (\tilde{A})^* \subseteq \tilde{A};\)
3. \(\vdash (\tilde{A} \cap \tilde{B})^* \equiv (\tilde{A})^* \cap (\tilde{B})^*;\)
4. \(\vdash (\tilde{A})^{**} \supseteq (\tilde{A})^*\).
**Lemma 2.9** (see [5]). Let \((X, \tau)\) be a fuzzifying topological space. For any \(\tilde{A} \in \mathcal{F}(X)\),

1. \(\vdash X \sim (\tilde{A})^{s} \equiv (X \sim \tilde{A})^{s}\);
2. \(\vdash X \sim (\tilde{A})^{s} \equiv (X \sim \tilde{A})^{s}\).

**Lemma 2.10** (see [2, 5]). If \([\tilde{A} \in \tilde{B}] = 1\), then

1. \(\vdash (\tilde{A})^{s} \subseteq (\tilde{B})^{s}\);
2. \(\vdash (\tilde{A})^{s} \subseteq (\tilde{B})^{s}\).

**Definition 2.11.** Let \((X, \tau)\) be a fuzzifying topological space.

1. The family of fuzzifying \(c\alpha\)-open [6] (resp., \(c\alpha\)-semi-open [5], \(c\beta\)-open [2]) sets, denoted by \(c\alpha\tau\) (resp., \(c\alpha\tau, cP\tau, c\beta\tau\) \(\in \mathcal{F}(P(X))\)), is defined as follows:
   \(A \in c\alpha\tau\) (resp., \(c\alpha\tau, cP\tau, c\beta\tau\)): \(\forall x (x \in A \cap A^{s} \subseteq (x \in A^{s} \subseteq (x \in A^{s}) \rightarrow (x \in A^{s})\).

2. The family of fuzzifying \(c\alpha\)-closed [6] (resp., \(c\alpha\)-semi-closed [5], \(c\beta\)-closed [2]) sets, denoted by \(c\alpha\tau\) (resp., \(c\alpha\tau, cP\tau, c\beta\tau\) \(\in \mathcal{F}(P(X))\)), is defined as follows:
   \(A \in c\alpha\tau\) (resp., \(c\alpha\tau, cP\tau, c\beta\tau\)): \(X \sim A \in c\alpha\tau\) (resp., \(c\alpha\tau, cP\tau, c\beta\tau\)).

**Definition 2.12** (see [10]). Let \((X, \tau)\) be a fuzzifying topological space.

1. The family of fuzzifying \(y\)-open sets, denoted by \(y\tau \in \mathcal{F}(P(X))\), is defined as follows:
   \(A \in y\tau := \forall x (x \in A \rightarrow x \in A^{s} \cup A^{s})\).

2. The family of fuzzifying \(y\)-closed sets, denoted by \(y\tau \in \mathcal{F}(P(X))\), is defined as follows:
   \(A \in y\tau := X \sim A \in y\tau\).

3. Let \((X, \tau), (Y, U)\) be two fuzzifying topological spaces. A unary fuzzy predicate \(\gamma C \in \mathcal{F}(Y^{X})\) called fuzzy \(\gamma\)-continuity, is given as \(\gamma C(f) := \forall u (u \in U \rightarrow f^{-1}(u) \in y\tau)\).

**Lemma 2.13** (see [7]). \(1) \vdash \tau \subseteq y\tau; (2) \vdash F \subseteq yF\).

**Definition 2.14** (see [10]). Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces. A unary fuzzy predicate \(C \in \mathcal{F}(Y^{X})\) called fuzzy continuity, is given as \(C(f) := \forall u (u \in U \rightarrow f^{-1}(u) \in \tau)\).

3. Fuzzifying \(c\gamma\)-open sets

**Definition 3.1.** Let \((X, \tau)\) be a fuzzifying topological space.

1. The family of fuzzifying \(c\gamma\)-open sets, denoted by \(c\gamma\tau \in \mathcal{F}(P(X))\), is defined as
   \(A \in c\gamma\tau := \forall x (x \in A \cap (A^{s} \cup A^{s}) \rightarrow A^{s})\).

2. The family of fuzzifying \(c\gamma\)-closed sets, denoted by \(c\gamma\tau \in \mathcal{F}(P(X))\), is defined as
   \(A \in c\gamma\tau := X \sim A \in c\gamma\tau\).

**Lemma 3.2.** For any \(\alpha, \beta, \gamma, \delta \in I, (1 - \alpha + \beta) \wedge (1 - \gamma + \delta) \leq 1 - (\alpha \wedge \gamma) + (\beta \wedge \delta)\).

**Theorem 3.3.** Let \((X, \tau)\) be a fuzzifying topological space, then

1. \(c\gamma\tau(X) = 1, c\gamma\tau(\emptyset) = 1\);
2. \(c\gamma\tau(A \cap B) \geq c\gamma\tau(A) \wedge c\gamma\tau(B)\).

**Proof.** The proof of (1) is straightforward.
(2) From Lemma 3.2, we have
\[ c\gamma \tau (A) \land c\gamma \tau (B) = \inf_{x \in A} (1 - (A^c \cup A^c))(x) + A^c(x)) \land \inf_{x \in B} (1 - (B^c \cup B^c))(x) + B^c(x)) \]
\[ = \inf_{x \in A \cap B} \left( (1 - (A^c \cup A^c))(x) + A^c(x)) \land (1 - (B^c \cup B^c))(x) + B^c(x)) \right) \]
\[ \leq \inf_{x \in A \cap B} \left( (A^c \cap B^c) \cap (A^c \cap B^c) \right)(x) + (A^c \cap B^c)(x)) \]
\[ \leq \inf_{x \in A \cap B} (1 - ((A \cap B)^c \cup (A \cap B)^c)) + (A \cap B)^c(x) = c\gamma \tau (A \cap B). \]

**Theorem 3.4.** Let \((X, \tau)\) be a fuzzifying topological space, then

1. \(c\gamma F(X) = 1, c\gamma F(\emptyset) = 1;\)
2. \(c\gamma F(A \cup B) \geq c\gamma F(A) \land c\gamma F(B).\)

**Proof.** From Theorem 3.3 the proof is obtained.

**Theorem 3.5.** Let \((X, \tau)\) be a fuzzifying topological space, then

1. (a) \(\tau \subseteq c\alpha \tau;\) (b) \(c\beta \tau \subseteq c\alpha \tau;\) (c) \(c\gamma \tau \subseteq c\alpha \tau;\) (d) \(c\gamma \tau \subseteq c\gamma \tau;\)
2. (a) \(\tau \subseteq c\alpha \tau;\) (b) \(c\beta \tau \subseteq c\alpha \tau;\) (c) \(c\gamma \tau \subseteq c\alpha \tau;\) (d) \(c\gamma F \subseteq c\alpha F;\) (e) \(c\gamma F \subseteq c\gamma F;\)
3. (a) \(\tau \subseteq c\alpha \tau;\) (b) \(c\beta \tau \subseteq c\alpha \tau;\) (c) \(c\gamma \tau \subseteq c\alpha \tau;\) (d) \(c\gamma F \subseteq c\alpha F;\) (e) \(c\gamma F \subseteq c\gamma F;\)
4. (a) \(\tau \subseteq c\alpha \tau;\) (b) \(c\beta \tau \subseteq c\alpha \tau;\) (c) \(c\gamma \tau \subseteq c\alpha \tau;\) (d) \(c\gamma F \subseteq c\alpha F;\) (e) \(c\gamma F \subseteq c\gamma F;\)
5. (a) \(\tau \subseteq c\alpha \tau;\) (b) \(c\beta \tau \subseteq c\alpha \tau;\) (c) \(c\gamma \tau \subseteq c\alpha \tau;\) (d) \(c\gamma F \subseteq c\alpha F;\) (e) \(c\gamma F \subseteq c\gamma F;\)

**Proof.** From the properties of the interior and closure operations and [9, Theorem 2.2(3)],

1. (a) \([A \in \tau] = [A \subseteq A^c] \subseteq [A \cap A^c = A^c] = [A \in c\alpha \tau];\)
2. (b) \([A \in c\beta \tau] = [A \cap A^c = A^c] \subseteq [A \cap A^c = A^c] = [A \in c\alpha \tau];\)
3. (c) \([A \in c\gamma \tau] = [A \cap A^c = A^c] \subseteq [A \cap A^c = A^c] = [A \in c\alpha \tau];\)
4. (d) \(c\gamma \tau (A) = \inf_{x \in A} (1 - \max(A^c(x), A^c(x)) + A^c(x)) \leq \inf_{x \in A} (1 - A^c(x) + A^c(x)) = c\gamma \tau (A);\)
5. (e) \(c\gamma \tau (A) = \inf_{x \in A} (1 - \max(A^c(x), A^c(x)) + A^c(x)) \leq \inf_{x \in A} (1 - A^c(x) + A^c(x)) = c\gamma \tau (A);\)
6. (f) \(c\beta \tau (A) = \inf_{x \in A} (1 - A^c(x) + A^c(x)) \leq \inf_{x \in A} (1 - \max(A^c(x), A^c(x)) + A^c(x)) = c\gamma \tau (A);\)
7. (g) \([A \in \tau] = [A \subseteq A^c] \subseteq [A \cap (A^c \cup A^c) \subseteq A^c] = [A \in c\gamma \tau].\)

(2) The proof is obtained from (1). \(\square\)

**Remark 3.6.** In crisp setting, that is, in case that the underlying fuzzifying topology is the ordinary topology, we have
\[ \vdash A \in \gamma \tau \land A \in c\gamma \tau \rightarrow A \in \tau. \] (3.2)

Of course the implication \(\rightarrow\) in (3.2) is either the Lukasiewicz’s implication or the Boolean’s implication since these implications are identical in crisp setting. But in fuzzifying setting the statement (3.2) may not be true as illustrated by the following counterexample.

**Counterexample 3.7.** Let \(X = \{a, b, c\}\) and let \(\tau\) be a fuzzifying topology on \(X\) defined as follows: \(\tau (X) = \tau (\emptyset) = \tau (\{a\}) = \tau (\{a, c\}) = 1; \tau (\{b\}) = \tau (\{a, b\}) = 0;\) and \(\tau (\{c\}) = \tau (\{b, c\}) = 1/8.\) From the definitions of the interior and the closure of a
subset of $X$ and the interior and the closure of a fuzzy set of $X$ we have the following: 
\[ y\tau((a,b)) = 7/8, \quad c_y\tau([a,b]) = 1/8. \]

**Theorem 3.8.** Let $(X, \tau)$ be a fuzzifying topological space. 
(1) $A \in \tau \iff (A \in y\tau \land A \in c_y\tau);$  
(2) if $[A \in y\tau] = 1$ or $[A \in c_y\tau] = 1$, then $\models A \in \tau \iff (A \in y\tau \land A \in c_y\tau)$.

**Proof.** (1) This follows from **Theorem 3.5**(g) and **Lemma 2.13**(1).

(2) Assume that $[A \in y\tau] = 1$, then for each $x \in A$, we have $\max(A^+(x), A^-(x)) = 1$ and so for each $x \in A$, $1 - \max(A^+(x), A^-(x)) + A^+(x) = A^+(x)$. Thus, $[A \in y\tau] \land [A \in c_y\tau] = [A \in c_y\tau] = \inf_{x \in A} (1 - \max(A^+(x), A^-(x)) + A^+(x)) = \inf_{x \in A} A^+(x) = [A \in \tau].$ Now, assume that $[A \in c_y\tau] = 1$, then for each $x \in A$, $1 - \max(A^+(x), A^-(x)) + A^+(x) = 1$ and so for each $x \in A$, $\max(A^+(x), A^-(x)) = A^+(x)$.

Thus,
\[
[A \in y\tau] \land [A \in c_y\tau] = [A \in y\tau] = \inf_{x \in A} \max(A^+(x), A^-(x)) = \inf_{x \in A} A^+(x) = [A \in \tau].
\]

**Theorem 3.9.** Let $(X, \tau)$ be a fuzzifying topological space. Then $\models (A \in y\tau \land A \in c_y\tau) \rightarrow A \in \tau$.

**Proof.**
\[
[A \in y\tau \land A \in c_y\tau] = \inf_{x \in A} \max(A^+(x), A^-(x)) \land \inf_{x \in A} (1 - \max(A^+(x), A^-(x)) + A^+(x))
\]
\[
= \max(0, \inf_{x \in A} \max(A^+(x), A^-(x)) + \inf_{x \in A} (1 - \max(A^+(x), A^-(x)) + A^+(x)) - 1)
\]
\[
\leq \inf_{x \in A} A^+(x) = [A \in \tau].
\]

(3.4)

**4. Fuzzifying $c_y$-neighborhood structure**

**Definition 4.1.** Let $x \in X$. The $c_y$-neighborhood system of $x$, denoted by $c_y N_x \in \mathcal{F}(P(X))$, is defined as $c_y N_x(A) = \sup_{x \in B \subseteq A} c_y \tau(B)$.

**Theorem 4.2.** A mapping $c_y N : X \rightarrow \mathcal{F}(P(X))$, $x \rightarrow c_y N_x$, where $\mathcal{F}(P(X))$ is the set of all normal fuzzy subsets of $P(X)$, has the following properties:

(1) $\models A \in c_y N_x \rightarrow x \in A;$  
(2) $\models A \subseteq B \rightarrow (A \in c_y N_x \land B \in c_y N_x);$  
(3) $\models A \in c_y N_x \land B \in c_y N_x \rightarrow A \cap B \in c_y N_x$.

Conversely, if a mapping $c_y N$ satisfies (2) and (3), then $c_y N$ assigns a fuzzifying topology on $X$ which is denoted by $\tau_{c_y N} \in \mathcal{F}(P(X))$ and defined as
\[
A \in \tau_{c_y N} := \forall x \ (x \in A \rightarrow A \in c_y N_x). \quad (4.1)
\]

**Proof.** (1) If $[A \in c_y N_x] = \sup_{x \in H \subseteq A} c_y \tau(H) > 0$, then there exists $H_x$ such that $x \in H_x \subseteq A$. Now, we have $[x \in A] = 1$. Therefore, $[A \in c_y N_x] \leq [x \in A]$ always holds.
(2) The proof is immediate.

(3) From Theorem 3.3(2), we have

\[ A \cap B \in c\gamma N_x \] = \( \sup_{x \in H \subseteq A \cap B} c\gamma \tau(H) = \sup_{x \in H_1 \subseteq A, x \in H_2 \subseteq B} c\gamma \tau(H_1 \cap H_2) \)

\[ \geq \sup_{x \in H_1 \subseteq A} c\gamma \tau(H_1) \land c\gamma \tau(H_2) \]

\[ = \sup_{x \in H_1 \subseteq A} c\gamma \tau(H_1) \land \sup_{x \in H_2 \subseteq B} c\gamma \tau(H_2) \]

\[ = [A \in c\gamma N_x \land B \in c\gamma N_x]. \] (4.2)

Conversely, we need to prove that \( \tau_{cyN} = \inf_{x \in A} c\gamma N_x(A) \) is a fuzzifying topology. From [8, Theorem 3.2] and since \( \tau_{cyN} \) satisfies properties (2) and (3), \( \tau_{cyN} \) is a fuzzifying topology.

\[ \text{THEOREM 4.3.} \quad \text{Let } (X, \tau) \text{ be a fuzzifying topological space. Then } \tau_{cyN} \subseteq \tau_{cyN}. \]

**Proof.** Let \( B \in P(X); \tau_{cyN}(B) = \inf_{x \in B} c\gamma N_x = \inf_{x \in B} \sup_{x \in A \subseteq B} c\gamma \tau(A) \geq c\gamma \tau(B). \]

\[ \text{5. Fuzzifying } c\gamma \text{-derived sets, fuzzifying } c\gamma \text{-closure, and fuzzifying } c\gamma \text{-interior} \]

**Definition 5.1.** Let \((X, \tau)\) be a fuzzifying topological space. The fuzzifying \( c\gamma \)-derived set of \( A \), denoted by \( c\gamma \)-d \( A \), is defined as

\[ c\gamma \text{-d}(A) = \inf_{B \cap (A - \{x\}) = \emptyset} (1 - c\gamma N_x(B)). \] (5.1)

**Lemma 5.2.** \( c\gamma \text{-d}(A)(x) = 1 - c\gamma N_x((X \sim A) \cup \{x\}). \)

**Proof.** From Theorem 4.2(2), we have

\[ c\gamma \text{-d}(A) = 1 - \sup_{B \cap (A - \{x\}) = \emptyset} c\gamma N_x(B) \]

\[ = 1 - \sup_{B \subseteq ((X - A) \cup \{x\})} c\gamma N_x(B) \]

\[ = 1 - c\gamma N_x((X \sim A) \cup \{x\}). \] (5.2)

**Theorem 5.3.** For any \( A, A \subseteq A \in F\tau_{cyN} \rightarrow c\gamma \text{-d}(A) \subseteq A. \)

**Proof.** From Lemma 5.2, we have

\[ [c\gamma \text{-d}(A) \subseteq A] = \inf_{x \in X - A} (1 - c\gamma \text{-d}(A)(x)) = \inf_{x \in X - A} c\gamma N_x((X \sim A) \cup \{x\}) \]

\[ = \inf_{x \in X - A} c\gamma N_x((X \sim A)) = [X \sim A \in \tau_{cyN}] = [A \in F\tau_{cyN}]. \] (5.3)

**Definition 5.4.** Let \((X, \tau)\) be a fuzzifying topological space. The \( c\gamma \)-closure of \( A \) is denoted and defined as follows: \( c\gamma \text{-cl}(A)(x) = \inf_{x \in B \supseteq A} (1 - c\gamma F(B)). \)
**Theorem 5.5.** (1) $cy\cdot\text{cl}(A)(x) = 1 - cyN_{x}(X \sim A)$;
(2) $\vdash cy\cdot\text{cl}(\emptyset) \equiv \emptyset$;
(3) $\vdash A \subseteq cy\cdot\text{cl}(A)$.

**Proof.** (1) $cy\cdot\text{cl}(A)(x) = \inf_{x \notin B\subseteq A}(1 - cyF(B)) = \inf_{x \in X - B\subseteq X - A}(1 - cy\tau(X \sim B)) = 1 - \sup_{x \in X - B\subseteq X - A}cy\tau(X \sim B) = 1 - cyN_{x}(X \sim A)$.
(2) $cy\cdot\text{cl}(\emptyset)(x) = 1 - cyN_{x}(X \sim \emptyset) = 0$.
(3) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $cyN_{x}(A) = 0$. If $x \in A$, then $cy\cdot\text{cl}(A)(x) = 1 - cyN_{x}(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq cy\cdot\text{cl}(A)] = 1$. □

**Theorem 5.6.** For any $x$ and $A$;
(1) $\vdash cy\cdot\text{cl}(A) \equiv cy\cdot d(A) \cup A$;
(2) $\vdash x \in cy\cdot\text{cl}(A) \rightarrow \forall B \ (B \in cyN_{x} \rightarrow A \cap B \neq \emptyset)$;
(3) $\vdash A \equiv cy\cdot\text{cl}(A) \rightarrow A \in F_{cyN}$.

**Proof.** (1) Applying Lemma 5.2 and Theorem 5.5(3), we have

$$x \in cy\cdot d(A) \cup A = \max(1 - cyN_{x}((X \sim A) \cup \{x\}), A(x)) = cy\cdot\text{cl}(A)(x). \tag{5.4}$$

(2) $[\forall B \ (B \in cyN_{x} \rightarrow A \cap B \neq \emptyset)] = \inf_{B \subseteq X - A}(1 - cyN_{x}(B)) = 1 - cyN_{x}(X \sim A) = [x \in cy\cdot\text{cl}(A)]$.
(3) From Theorem 5.5(1), we have

$$A \equiv cy\cdot\text{cl}(A) = \inf_{x \in X - A} (1 - cy\cdot\text{cl}(A)(x))$$

$$= \inf_{x \in X - A} cyN_{x}(X \sim A) = [(X \sim A) \in F_{cyN}] = [A \in \tau_{cyN}]. \tag{5.5}$$

□

**Theorem 5.7.** For any $A$ and $B$, $\vdash B \equiv cy\cdot\text{cl}(A) \rightarrow B \in F_{cyN}$.

**Proof.** If $[A \subseteq B] = 0$, then $[B \equiv cy\cdot\text{cl}(A)] = 0$. Now, we suppose $[A \subseteq B] = 1$, then we have $[B \subseteq cy\cdot\text{cl}(A)] = 1 - \sup_{x \in B - A} cyN_{x}(X \sim A)$ and $[cy\cdot\text{cl}(A) \subseteq B] = \inf_{x \in X - B} cyN_{x}(X \sim A)$. So,

$$[B \equiv cy\cdot\text{cl}(A)] = \max \left(0, \inf_{x \notin B} cyN_{x}(X \sim A) - \sup_{x \in B - A} cyN_{x}(X \sim A) \right). \tag{5.6}$$

If $[B \equiv cy\cdot\text{cl}(A)] > t$, then $\inf_{x \notin B} cyN_{x}(X \sim A) > t + \sup_{x \in B - A} cyN_{x}(X \sim A)$. For any $x \in X - B$, $\sup_{x \in C \subseteq X - A} cy\tau(C) > t + \sup_{x \in B - A} cyN_{x}(X \sim A)$, that is, there exists $C_{x}$ such that $x \in C_{x} \subseteq X - A$ and $cy\tau(C_{x}) > t + \sup_{x \in B - A} cyN_{x}(X \sim A)$. Now, we want to prove that $C_{x} \subseteq X - B$. If not, then there exists $x' \in B \sim A$ such that $x' \in C_{x}$. Hence, we can obtain that $\sup_{x \in B - A} cyN_{x}(X \sim A) \geq cyN_{x'}(X \sim A) \geq cy\tau(C_{x}) > t + \sup_{x \in B - A} cyN_{x}(X \sim A)$. This is a contradiction. Therefore, $F_{cyN}(B) = \tau_{cyN}(X \sim B) = \inf_{x \notin B} cyN_{x}(X \sim B) \geq \inf_{x \notin B} cy\tau(C_{x}) > t + \sup_{x \in B - A} cyN_{x}(X \sim A)$. Since $t$ is arbitrary, it holds that $[B \equiv cy\cdot\text{cl}(A)] \leq [B \in F_{cyN}]$. □

**Definition 5.8.** Let $(X, \tau)$ be a fuzzifying topological space. For any $A \subseteq X$, the $cy$-interior of $A$ is given as follows: $cy\cdot\text{int}(A)(x) = cyN_{x}(A)$. 

THEOREM 5.9. For any \( x, A, \) and \( B, \)

1. \( \forall B \in \tau_{cyN} \land B \subseteq A \rightarrow B \subseteq cy\text{-}int(A); \)
2. \( \forall A \equiv cy\text{-}int(A) \rightarrow A \in \tau_{cyN}; \)
3. \( \forall x \in cy\text{-}int(A) \rightarrow x \in A \land x \in (X \sim cy\text{-}d(X \sim A)); \)
4. \( \forall cy\text{-}int(A) \equiv X \sim cy\text{-}cl(X \sim A); \)
5. \( \forall B \equiv cy\text{-}int(A) \rightarrow B \in \tau_{cyN}; \)
6. \( \forall cy\text{-}int(X) \equiv X, \) \( \forall cy\text{-}int(A) \subseteq A. \)

PROOF. (1) If \( B \not\subseteq A, \) then \( B \in \tau_{cyN} \land B \subseteq A = 0. \) If \( B \subseteq A, \) then

\[
[B \subseteq cy\text{-}int(A)] = \inf_{x \in B} cy\text{-}int(A)(x) = \inf_{x \in B} cyN_x(A) \geq \inf_{x \in B} cyN_x(B) \]

\[
= [B \in \tau_{cyN}] = [B \in \tau_{cyN} \land B \subseteq A]. \tag{5.7}
\]

(2)

\[
[A \equiv cy\text{-}int(A)] = \min \left( \inf_{x \in A} cy\text{-}int(A)(x), \inf_{x \in X \setminus A} \left( 1 - cy\text{-}int(A)(x) \right) \right) \]

\[
= \inf_{x \in A} cy\text{-}int(A)(x) = \inf_{x \in A} cyN_x(A) = [A \in \tau_{cyN}]. \tag{5.8}
\]

(3) If \( x \notin A, \) then \( [x \in cy\text{-}int(A)] = 0 = [x \in A \land x \in (X \sim cy\text{-}d(X \sim A))]. \) If \( x \in A, \) then \( [x \in cy\text{-}d(X \sim A)] = 1 - cyN_x(A \cup \{x\}) = 1 - cyN_x(A) = 1 - cy\text{-}int(A)(x), \) so that \( \{x \in A \land x \in (X \sim cy\text{-}d(X \sim A))\} = [x \in cy\text{-}int(A)]. \)

(4) It follows from Theorem 5.5(1).

(5) From (4) and Theorem 5.7, we have

\[
[B \equiv cy\text{-}int(A)] = [X \sim B \equiv cy\text{-}cl(X \sim A)] \leq [X \sim B \in F\tau_{cyN}] = [B \in \tau_{cyN}]. \tag{5.9}
\]

(6) (a) It is obtained from (4) above and from Theorem 5.5(2).

(b) It is obtained from (3) above. \qed

6. Fuzzifying cy-continuous functions

DEFINITION 6.1. Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces. For any \( f \in Y^X, \) a unary fuzzy predicates \( cyC \in \mathcal{F}(Y^X), \) called cy-continuity, is given as

\[
cyC(f) := \forall u \ (u \in U - f^{-1}(u) \in cy\tau). \tag{6.1}
\]

DEFINITION 6.2. Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces. For any \( f \in Y^X, \) we define the unary fuzzy predicates \( y_j \in \mathcal{F}(Y^X) \) where \( j = 1, 2, \ldots, 5 \) as follows:

1. \( y_1(f) := \forall B (B \in F_Y - f^{-1}(B) \in cyF_X), \) where \( F_Y \) is the family of closed subsets of \( Y \) and \( cyF_X \) is the family of \( cy\)-closed subsets of \( X; \)
2. \( y_2(f) := \forall x \forall u \ (u \in N_{f(x)} - f^{-1}(u) \in cyN_x), \) where \( N \) is the neighborhood system of \( Y \) and \( cyN \) is the \( cy\)-neighborhood system of \( X; \)
3. \( y_3(f) := \forall x \forall u \ (u \in N_{f(x)} - \exists v (f(v) \subseteq u - v \in cyN_x)); \)
4. \( y_4(f) := \forall A (f(cy\text{-}cl_X(A)) \subseteq cl_Y(f(A))); \)
(5) \( y_5(f) := \forall B (c_y \cdot \text{cl}_X (f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y (B))). \)

**THEOREM 6.3.**

(1) \( f \in c_y C \Leftrightarrow f \in y_1; \)

(2) \( \Rightarrow f \in c_y C \Rightarrow f \in y_2; \)

(3) \( \Rightarrow f \in y_2 \Rightarrow f \in y_j \text{ for } j = 3, 4, 5. \)

**PROOF.**

(1) We prove that \([f \in c_y C] = [f \in y_1]\)

\[
[f \in y_1] = \inf_{F \in P(Y)} \min (1, 1 - F_Y(F) + c_y F_X (f^{-1}(F)))
\]

\[= \inf_{F \in P(Y)} \min (1, 1 - U(Y - F) + c_y \tau (X \sim f^{-1}(F)))
\]

\[= \inf_{u \in P(Y)} \min (1, 1 - U(u) + c_y \tau (f^{-1}(u)))
\]

\[= [f \in c_y C]. \]

(2) We prove that \( y_2(f) \geq c_y C(f) \). If \( N_{f(x)}(u) \leq c_y N_X (f^{-1}(u)) \), the result holds. Suppose \( N_{f(x)}(u) > c_y N_X (f^{-1}(u)) \). It is clear that if \( f(x) \in A \subseteq u \) then \( x \in f^{-1}(A) \subseteq f^{-1}(u) \). Then,

\[
N_{f(x)}(u) - c_y N_X (f^{-1}(u)) = \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} c_y \tau(B)
\]

\[\leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} c_y \tau (f^{-1}(A)) \]  \hspace{1cm} (6.3)

So,

\[1 - N_{f(x)}(u) + c_y N_X (f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u} (1 - U(A) + c_y \tau (f^{-1}(A))) \text{ and thus} \]

\[
\inf_{f(x) \in A \subseteq u} (1 - N_{f(x)}(u) + c_y N_X (f^{-1}(u))) \geq \inf_{f(x) \in A \subseteq u} (1 - U(A)) + c_y \tau (f^{-1}(A)))
\]

\[\geq \inf_{v \in P(Y)} \min (1, 1 - U(v) + c_y \tau (f^{-1}(v))) \]  \hspace{1cm} (6.4)

\[= c_y C(f). \]

Hence, \( \inf_{x \in X} \inf_{u \in P(Y)} \min (1, 1 - N_{f(x)}(u) + c_y N_X (f^{-1}(u))) \geq [f \in c_y C]. \)

(3) (a) We prove that \( \Rightarrow f \in y_2 \Rightarrow f \in y_3 \). Since \( c_y N_X \) is monotonic (Theorem 4.2(2)), it is clear that \( \sup_{v \in P(X), f(v) \subseteq u} c_y N_X (v) = \sup_{v \in P(X), f(v) \subseteq f^{-1}(u)} c_y N_X (v) = c_y N_X (f^{-1}(u)). \)

Then,

\[y_3(f) = \inf_{x \in X} \inf_{u \in P(Y)} \min (1, 1 - N_{f(x)}(u) + c_y N_X (v)) \]

\[= \inf_{x \in X} \inf_{u \in P(Y)} \min (1, 1 - N_{f(x)}(u) + c_y N_X (f^{-1}(u))) = y_2(f). \]  \hspace{1cm} (6.5)

(b) We prove that \( \Rightarrow f \in y_4 \Rightarrow f \in y_5. \)
First, for each \( B \in P(Y) \), there exists \( A \in P(X) \) such that \( f^{-1}(B) = A \) and \( f(A) \subseteq B \). So, \([\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \geq [\gamma \cdot \text{cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] \). Hence,

\[
\gamma_5(f) = \inf_{B \in P(Y)} [\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \\
\geq \inf_{A \in P(X)} [\gamma \cdot \text{cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] = y_4(f). \quad (6.6)
\]

Second, for each \( A \in P(X) \), there exists \( B \in P(Y) \) such that \( f(A) = B \) and \( f^{-1}(B) \supseteq A \). Hence, \([\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \leq [\gamma \cdot \text{cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] \). Thus,

\[
y_4(f) = \inf_{A \in P(X)} [\gamma \cdot \text{cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] \\
\geq \inf_{B \in P(Y), B = f(A)} [\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \\
\geq \inf_{B \in P(Y)} [\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] = y_5(f). \quad (6.7)
\]

(c) We prove that \( f \in y_5 \Leftrightarrow f \in y_2 \); from Theorem 5.5(1),

\[
y_5(f) = \forall B(\gamma \cdot \text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))) \\
= \inf_{B \in P(Y)} \inf_{x \in X} \min(1, 1 - (1 - \gamma N_x(X \sim f^{-1}(B))) + 1 - N_{f(x)}(Y \sim B)) \\
= \inf_{B \in P(Y)} \inf_{x \in X} \min(1, 1 - N_{f(x)}(Y \sim B) + \gamma N_x(X \sim f^{-1}(B))) \\
= \inf_{u \in P(Y)} \inf_{x \in X} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) = y_2(f). \quad \square
\]

**Remark 6.4.** In the following theorem, we indicate the fuzzifying topologies with respect to which we evaluate the degree to which \( f \) is continuous or \( \gamma C \)-continuous. Thus, the symbols \((\tau, U)\cdot C(f)\), \((\tau_{\gamma \cdot N}, U)\cdot C(f)\), \((\tau, U_{\gamma \cdot N})\cdot \gamma C(f)\), and so forth, will be understood.

Applying Theorems 3.5(g) and 4.3, one can deduce the following theorem.

**Theorem 6.5.** (1) \( f \in (\tau, U_{\gamma \cdot N})\cdot C \rightarrow f \in (\tau, U)\cdot C \);

(2) \( f \in (\tau, U)\cdot \gamma C \rightarrow f \in (\tau_{\gamma \cdot N}, U)\cdot C \);

(3) \( f \in (\tau, U)\cdot C \rightarrow f \in (\tau, U)\cdot \gamma C \).

7. Decompositions of fuzzy continuity in fuzzifying topology

**Theorem 7.1.** Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces. For any \( f \in Y^X \),

\[
\Rightarrow C(f) \rightarrow (\gamma C(f) \land \gamma C(f)). \quad (7.1)
\]

**Proof.** The proof is obtained from Lemma 2.13(1) and Theorem 3.5(g). \( \square \)

**Remark 7.2.** In crisp setting, that is, in the case that the underlying fuzzifying topology is the ordinary topology, one can have \( \Rightarrow (\gamma C(f) \land \gamma C(f)) \rightarrow C(f) \).
But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

**Counterexample 7.3.** Let \((X, \tau)\) be the fuzzifying topological space defined in **Counterexample 3.7**. Consider the identity function \(f\) from \((X, \tau)\) onto \((X, \sigma)\), where \(\sigma\) is a fuzzifying topology on \(X\) defined as follows:

\[
\sigma(A) = \begin{cases} 
1, & A \in \{X, \emptyset, \{a, b\}\}, \\
0, & \text{otherwise.}
\end{cases}
\]

(7.2)

Then, \(7/8 \wedge 1/8 = \gamma C(f) \wedge c \gamma C(f) \not\leq C(f) = 0\).

**Theorem 7.4.** Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces. For any \(f \in Y^X\),

\[
\Rightarrow C(f) \rightarrow (\gamma C(f) \longleftrightarrow c \gamma C(f)).
\]

(7.3)

**Proof.** \([\gamma C(f) \rightarrow c \gamma C(f)] = \min(1, 1 - \gamma C(f) + c \gamma C(f)) \geq \gamma C(f) \wedge c \gamma C(f)\). Also, \([c \gamma C(f) \rightarrow \gamma C(f)] = \min(1, 1 - c \gamma C(f) + \gamma C(f)) \geq \gamma C(f) \wedge c \gamma C(f)\). Then from **Theorem 7.1** we have \([\gamma C(f) \wedge c \gamma C(f)] \geq C(f)\) and so the result holds.

\[\blacksquare\]

**Theorem 7.5.** Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces and let \(f \in Y^X\). If \([\gamma \tau(f^{-1}(u))] = 1\) or \([c \gamma \tau(f^{-1}(u))] = 1\) for each \(u \in P(Y)\), then \(\Rightarrow C(f) \rightarrow (\gamma C(f) \wedge c \gamma C(f))\).

**Proof.** We need to prove that \(C(f) = \gamma C(f) \wedge c \gamma C(f)\). Applying **Theorem 3.8**(2), we have

\[
y C(f) \wedge c y C(f) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + y \tau(f^{-1}(u))) \wedge \inf_{u \in P(Y)} \min(1, 1 - U(u) + c y \tau(f^{-1}(u))) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + c y \tau(f^{-1}(u))) \wedge (1 - U(u) + c y \tau(f^{-1}(u)))) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + (y \tau(f^{-1}(u)) \wedge c y \tau(f^{-1}(u)))) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \tau(f^{-1}(u))) = C(f).
\]

(7.4)

\[\blacksquare\]

**Theorem 7.6.** Let \((X, \tau)\) and \((Y, U)\) be two fuzzifying topological spaces and let \(f \in Y^X\),

1. If \([\gamma \tau(f^{-1}(u))] = 1\) for each \(u \in P(Y)\), then \(\Rightarrow y C(f) \rightarrow (c y C(f) \rightarrow C(f))\),
2. If \([c \gamma \tau(f^{-1}(u))] = 1\) for each \(u \in P(Y)\), then \(\Rightarrow c y C(f) \rightarrow (y C(f) \rightarrow C(f))\).

**Proof.** (1) Since \([\gamma \tau(f^{-1}(u))] = 1\) and so \([f^{-1}(u) \subseteq ((f^{-1}(u)) \cap (f^{-1}(u))^{-})] = 1\), then \([f^{-1}(u) \cap ((f^{-1}(u)) \cap (f^{-1}(u))^{-})] = (f^{-1}(u)) \cap ((f^{-1}(u)) \cap (f^{-1}(u))^{-}) = [f^{-1}(u) \subseteq (f^{-1}(u)) \cap (f^{-1}(u))^{-}] = [f^{-1}(u) \subseteq (f^{-1}(u)) \cap (f^{-1}(u))^{-}] = [f^{-1}(u) \subseteq f^{-1}(u)]\).
Thus,
\[ c\gamma C(f) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + c\gamma \tau(f^{-1}(u))) \]
\[ = \inf_{u \in P(Y)} \min(1, 1 - U(u) + [(f^{-1}(u) \cap (f^{-1}(u))^\circ \cup (f^{-1}(u))^\circ) \subseteq (f^{-1}(u))^\circ)] \]
\[ = \inf_{u \in P(Y)} \min(1, 1 - U(u) + [f^{-1}(u) \subseteq (f^{-1}(u))^\circ]) \]
\[ = \inf_{u \in P(Y)} \min(1, 1 - U(u) + \tau(f^{-1}(u))) = C(f). \]  
(7.5)

(2) Since \([c\gamma \tau(f^{-1}(u))] = 1\), one can deduce that \((f^{-1}(u))^\circ \cup (f^{-1}(u))^\circ = (f^{-1}(u))^\circ\). So,
\[
yC(f) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + y\tau(f^{-1}(u))) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + [(f^{-1}(u) \cap ((f^{-1}(u))^\circ \cup (f^{-1}(u))^\circ)) \subseteq (f^{-1}(u))^\circ)] \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + [f^{-1}(u) \subseteq (f^{-1}(u))^\circ]) \\
= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \tau(f^{-1}(u))) = C(f). \]  
(7.6)

**Theorem 7.7.** Let \((X, \tau), (Y, U), \) and \((Z, V)\) be three fuzzifying topological spaces. For any \(f \in Y^X\) and \(g \in Z^Y\),

1. \(\vdash c\gamma C(f) \rightarrow (C(g) \rightarrow c\gamma C(g \circ f))\);
2. \(\vdash C(g) \rightarrow (c\gamma C(g) \rightarrow c\gamma C(g \circ f)).\)

**Proof.** (1) We prove that \([c\gamma C(f)] \leq [(C(g) \rightarrow c\gamma C(g \circ f))].\) If \([C(g)] \leq [c\gamma C(g \circ f)],\) then the result holds. If \([C(g)] > [c\gamma C(g \circ f)],\) then we have
\[
[C(g)] - [c\gamma C(g \circ f)] = \inf_{v \in P(Z)} \min(1, 1 - V(v) + U(g^{-1}(v))) \\
- \inf_{v \in P(Z)} \min(1, 1 - V(v) + c\gamma \tau(g \circ f)^{-1}(v)) \\
\leq \sup_{v \in P(Z)} (U(g^{-1}(v)) - c\gamma \tau(g \circ f)^{-1}(v)) \]  
(7.7)
\[
\leq \sup_{u \in P(Y)} (U(u) - c\gamma \tau(f^{-1}(u))).
\]
Therefore,
\[
[C(g) \rightarrow c\gamma C(g \circ f)] = \min(1, 1 - [C(g)] + [c\gamma C(g \circ f)]) \\
\geq \inf_{u \in P(Y)} \min(1, 1 - U(u) + c\gamma \tau(f^{-1}(u))) = c\gamma C(f). \]
(7.8)

(2) Since the conjunction \(\wedge\) is commutative, from (1) above, one can deduce that
\[
[C(g) \rightarrow (c\gamma C(f) \rightarrow c\gamma C(g \circ f))] = [\neg(C(g) \wedge c\gamma C(f) \wedge \neg c\gamma C(g \circ f))] \\
= [\neg(c\gamma C(f) \wedge C(g) \wedge \neg c\gamma C(g \circ f))] \]  
(7.9)
\[
= [c\gamma C(f) \rightarrow (C(g) \rightarrow c\gamma C(g \circ f))] = 1. \]

\(\square\)
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