THE GALOIS ALGEBRAS AND THE AZUMAYA GALOIS EXTENSIONS

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Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$, $C$ the center of $B$, $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$, $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in K$, and $B_K = (\oplus_{g \in K} J_g)$. Then $B_K$ is a central weakly Galois algebra with Galois group induced by $K$. Moreover, an Azumaya Galois extension $B$ with Galois group $K$ is characterized by using $B_K$.

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1. Introduction. Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$ and $C$ the center of $B$. The class of Galois algebras has been investigated by DeMeyer [2], Kanzaki [6], Harada [4, 5], and the authors [7]. In [2], it was shown that if $R$ contains no idempotents but 0 and 1, then $B$ is a central Galois algebra with Galois group $K$ and $C$ is a commutative Galois algebra with Galois group $G/K$ where $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$ [2, Theorem 1]. This fact was extended to the Galois algebra $B$ over $R$ containing more than two idempotents [6, Proposition 3], and generalized to any Galois algebra $B$ [7, Theorem 3.8] by using the Boolean algebra $B_\alpha$ generated by $\{ 0, e_g \mid g \in G \text{ for a central idempotent } e_g \}$ where $BJ_g = Be_g$ and $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$ [6]. The purpose of this paper is to show that there exists a subalgebra $B_K$ of $B$ such that $B_K$ is a central weakly Galois algebra with Galois group $K|B_K$ induced by $K$ where a weakly Galois algebra was defined in [8] and that $B_K B^K$ is an Azumaya weakly Galois extension with Galois group $K|B_K$ where an Azumaya Galois extension was studied in [1]. Thus some characterizations of an Azumaya Galois extension $B$ of $B^K$ with Galois group $K$ are obtained, and the results as given in [2, 6] are generalized.

2. Definitions and notations. Throughout, let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$, $C$ the center of $B$, and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$. We keep the definitions of a Galois extension, a Galois algebra, a central Galois algebra, a separable extension, and an Azumaya algebra as defined in [7]. An Azumaya Galois extension $A$ with Galois group $G$ is a Galois extension $A$ of $A^G$ which is a $C^G$-Azumaya algebra where $C$ the center of $A$ [1]. A weakly Galois extension $A$ with Galois group $G$ is a finitely generated projective left module $A$ over $A^G$ such that $A_1G \cong \text{Hom}_{A^G}(A, A)$ where $A_1 = \{ a_1 \}$, a left multiplication map by $a \in A$ [8]. We call that $A$ is a weakly Galois algebra with Galois group $G$ if $A$ is a weakly Galois extension with Galois group $G$ such that $A^G$ is contained in the center of $A$ and that
A is a central weakly Galois algebra with Galois group $G$ if $A$ is a weakly Galois extension with Galois group $G$ such that $A^G$ is the center of $A$. An Azumaya weakly Galois extension $A$ with Galois group $G$ is a weakly Galois extension $A$ of $A^G$ which is a $C^G$-Azumaya algebra where $C$ is the center of $A$.

3. A weakly Galois algebra. In this section, let $B$ be a Galois algebra over $R$ with Galois group $G$, $C$ the center of $B$, $B^G = \{ b \in B \mid g(b) = b \text{ for all } g \in G \}$, and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$. Then, $B = \oplus \sum_{g \in G} J_g = (\oplus \sum_{g \in K} J_g) \oplus (\oplus \sum_{g \notin K} J_g)$ where $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ [6, Theorem 1]. We denote $\oplus \sum_{g \in K} J_g$ by $B_K$ and the center of $B_K$ by $Z$. Clearly, $K$ is a normal subgroup of $G$. We show that $B_K$ is an Azumaya algebra over $Z$ and a central weakly Galois algebra with Galois group $K|B_K$.

**Theorem 3.1.** The algebra $B_K$ is an Azumaya algebra over $Z$.

**Proof.** By the definition of $B_K$, $B_K = \oplus \sum_{g \in K} J_g$, so $C(= J_1) \subset B_K$. Since $B$ is a Galois algebra with Galois group $G$ and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$, the order of $K$ is a unit in $C$ by [6, Proposition 5]. Moreover, $K$ is an $C$-automorphism group of $B$, so $B_K$ is a $C$-separable algebra by [5, Proposition 5]. Thus $B_K$ is an Azumaya algebra over $Z$.

In order to show that $B_K$ is a central weakly Galois algebra with Galois group $K|B_K$, we need two lemmas.

**Lemma 3.2.** Let $L = \{ g \in K \mid g(a) = a \text{ for all } a \in B_K \}$. Then, $L$ is a normal subgroup of $K$ such that $K(= K/L)$ is an automorphism group of $B_K$ induced by $K$ (i.e., $K|B_K \cong K$).

**Proof.** Clearly, $L$ is a normal subgroup of $K$, so for any $h \in K$,

$$h(B_K) = \oplus \sum_{g \in K} h(J_g) = \oplus \sum_{g \in K} J_{gh^{-1}} = \oplus \sum_{g \in hKh^{-1}} J_g = \oplus \sum_{g \in K} J_g = B_K. \quad (3.1)$$

Thus $K|B_K \cong K$. 

**Lemma 3.3.** The fixed ring of $B_K$ under $K$, $(B_K)^K = Z$.

**Proof.** Let $x$ be any element in $(B_K)^K$ and $b$ any element in $B_K$. Then $b = \sum_{g \in K} b_g$ where $b_g \in J_g$ for each $g \in K$. Hence $bx = \sum_{g \in K} b_g x = \sum_{g \in K} g(x)b_g = \sum_{g \in K} x b_g = x \sum_{g \in K} b_g = xb$. Therefore $x \in Z$. Thus $(B_K)^K \subset Z$. Conversely, for any $z \in Z$ and $g \in K$, we have that $zx = xz = g(z)x$ for any $x \in J_g$, so $(g(z) - z)x = 0$ for any $x \in J_g$. Hence $(g(z) - z)J_g = \{0\}$. Noting that $BJ_g = J_g B = B$, we have that $(g(z) - z)B = \{0\}$, so $g(z) = z$ for any $z \in Z$ and $g \in K$. Thus $Z \subset (B_K)^K$. Therefore $(B_K)^K = Z$.

**Theorem 3.4.** The algebra $B_K$ is a central weakly Galois algebra with Galois group $K|B_K \cong K$.

**Proof.** By Lemma 3.3, it suffices to show that (1) $B_K$ is a finitely generated projective module over $Z$, and (2) $(B_K)|K \cong \text{Hom}_Z(B_K, B_K)$. Part (1) is a consequence of Theorem 3.1. For part (2), since $B_K$ is an Azumaya algebra over $Z$ by Theorem 3.1 again, $B_K \otimes_Z B_K^0 \cong \text{Hom}_Z(B_K, B_K)$ [3, Theorem 3.4, page 52] by extending the map $(a \otimes b)(x) = axb$ linearly for $a \otimes b \in B_K \otimes_Z B_K^0$ and each $x \in B_K$ where $B_K^0$ is the
opposite algebra of $B_K$. By denoting the left multiplication map with $a \in B_K$ by $a_l$ and the right multiplication map with $b \in B_K$ by $b_r$, $(a \otimes b)(x) = (a_l b_r)(x) = axb$. Since $B_K = \bigoplus_{g \in K} J_g$, $B_K \otimes \mathcal{B}_K = \bigoplus_{g \in K} (B_K)_l (J_g)_r$. Observing that $(J_g)_r = (J_g)_{\mathcal{B}}^{-1}$ where $\mathcal{B} = g \mid b \in B_K \in K \mid B_K \cong K$, we have that $B_K \otimes \mathcal{B}_K = \bigoplus_{g \in K} (B_K)_l (J_g)_r = \bigoplus_{g \in K} (B_K)_l (g)_{\mathcal{B}}^{-1} = \bigoplus_{g \in K} (B_K)_l (B_K)_{\mathcal{B}}^{-1}$.

Moreover, since $BI_B = B$ for each $g \in K$ and $B = \bigoplus_{h \in G} J_h = B \otimes (\bigoplus_{h \in G} J_h)$, $B_K \otimes (\bigoplus_{h \in G} J_h) = B = B_I_B = B \otimes (\bigoplus_{h \in G} J_h)_{B_K}$ such that $B_K J_g \subset B_K$ and $\bigoplus_{h \in G} J_h = \bigoplus_{h \in G} J_h$. Hence $B_K J_g \subset B$ for each $g \in K$. Therefore $B_K \otimes \mathcal{B}_K = \bigoplus_{g \in K} (B_K)_l (B_K)_{\mathcal{B}}^{-1} = \bigoplus_{g \in K} (B_K)_{\mathcal{B}}^{-1} = (B_K)_l K$. Thus $(B_K)_l K \cong \text{Hom}_Z(B_K, B_K)$. This completes the proof of part (2). Thus $B_K$ is an Azumaya algebra over Galois group $K \mid B_K \cong K$.

Recall that an algebra $A$ is called an Azumaya weakly Galois extension of $A^K$ with Galois group $K$ if $A$ is a weakly Galois extension of $A^K$ which is a $C^K$-Azumaya algebra where $C$ is the center of $A$. Next, we show that $B_K$ is an Azumaya weakly Galois extension with Galois group $K \mid B_K \cong K$. We begin with the following two lemmas about $B_K$.

**Lemma 3.5.** *The fixed ring of $B$ under $K$, $B^K = V_B(B_K)$.*

**Proof.** For any $b \in B^K$ and $x \in J_g$ for any $g \in K$, we have that $xb = g(b)x = bx$, so $b \in V_B(B_K)$ for any $g \in K$. Thus $b \in V_B(B_K)$. Conversely, for any $b \in V_B(B_K)$ and $g \in K$, we have that $bx = xb = g(b)x$ for any $x \in J_g$, so $(g(b) - b)x = 0$ for any $x \in J_g$. Hence $(g(b) - b)J_g = \{0\}$. But $BI_B = B$ for any $g \in K$, so $(g(b) - b)B = \{0\}$. Thus $g(b) = b$ for any $g \in K$; and so $b \in B^K$. Therefore $B^K = V_B(B_K)$.

**Lemma 3.6.** *The algebra $B^K$ is an Azumaya algebra over $Z$ where $Z$ is the center of $B_K$.***

**Proof.** Since $B$ is a Galois algebra over $R$ with Galois group $G$, $B$ is an Azumaya algebra over its center $C$. By the proof of *Theorem 3.1*, $B_K$ is a $C$-separable subalgebra of $B$, so $V_B(B_K)$ is a $C$-separable subalgebra of $B$ and $V_B(B_K) = B_K$ by the commutator theorem for Azumaya algebras [3, Theorem 4.3, page 57]. This implies that $B_K$ and $V_B(B_K)$ have the same center $Z$. Thus $V_B(B_K)$ is an Azumaya algebra over $Z$. But, by *Lemma 3.5*, $B^K = V_B(B_K)$, so $B^K$ is an Azumaya algebra over $Z$.

**Theorem 3.7.** *Let $A = B_KB^K$. Then $A$ is an Azumaya weakly Galois extension with Galois group $K \mid A \cong K$.***

**Proof.** Since $B_K$ is a central weakly Galois algebra with Galois group $K \mid B_K \cong K$ by *Theorem 3.4*, $B_K$ is a finitely generated projective module over $Z$ and $(B_K)_l K \cong \text{Hom}_Z(B_K, B_K)$. By *Lemma 3.6*, $B^K$ is an Azumaya algebra over $Z$, so $A(\cong B_K \otimes Z B^K)$ is a finitely generated projective module over $B^K(= A^K)$. Moreover, since $B^K = V_B(B_K)$ by *Lemma 3.5* and $(B_K)_l K \cong \text{Hom}_Z(B_K, B_K)$,

$$A_l K = (B_K B^K)_l K = (B_K)_l K(B^K)_r \cong B_K K \otimes Z B^K \cong \text{Hom}_Z(B_K, B_K) \otimes Z B^K \cong \text{Hom}_Z(B_K B^K, B_K B^K) \cong \text{Hom}_Z(B_K B^K, B_K B^K)$$

$$= \text{Hom}_Z(A, A).$$
Thus \( A \) is a weakly Galois extension of \( A^K \) with Galois group \( K|_A \cong \bar{K} \). Next, we claim that \( A \) has center \( Z \) and \( A^K \) is an Azumaya algebra over \( Z^K \). In fact, \( B_K \) and \( B^K \) are Azumaya algebras over \( Z \) by Theorem 3.1 and Lemma 3.6, respectively, so \( A (= B_K B^K) \) has center \( Z \) and \( A^K = (B_K B^K)^K = B^K \). Noting that \( B^K \) is an Azumaya algebra over \( Z \), we conclude that \( A^K \) is an Azumaya algebra over \( Z^K \). Thus \( A \) is an Azumaya weakly Galois extension with Galois group \( K|_A \cong \bar{K} \).

4. An Azumaya Galois extension. In this section, we give several characterizations of an Azumaya Galois extension \( B \) by using \( B_K \). This generalizes the results in [2, 6]. The \( Z \)-module \( \{ b \in B_K \mid bx = g(x)b \text{ for all } x \in B_K \} \) is denoted by \( J^{(B_K)}_\bar{g} \) for each \( \bar{g} \in \bar{K} \) where \( \bar{K} (= K/L) \) is defined in Lemma 3.2.

\textbf{Lemma 4.1.} The algebra \( B_K \) is a central Galois algebra with Galois group \( K|_{B_K} \equiv \bar{K} \) if and only if \( J^{(B_K)}_\bar{g} = \oplus \{ l \in L \mid J^{(B_K)}_\bar{g} \subset J^{(B_K)}_l \} \) for each \( \bar{g} \in \bar{K} \).

\textbf{Proof.} Let \( B_K \) be a central Galois algebra with Galois group \( K|_{B_K} \equiv \bar{K} \). Then \( B_K = \oplus \sum_{\bar{g} \in \bar{K}} J^{(B_K)}_\bar{g} \) [6, Theorem 1]. Next it is easy to check that \( \oplus \sum_{l \in L} J^g_l \subset J^{(B_K)}_\bar{g} \). But \( B_K = \oplus \sum_{\bar{g} \in \bar{K}} J^{(B_K)}_\bar{g} \), so \( \oplus \sum_{\bar{g} \in \bar{K}} J^g_{\bar{g}} = \oplus \sum_{\bar{g} \in \bar{K}} J^{(B_K)}_\bar{g} \) where \( \oplus \sum_{l \in L} J^g_l \subset J^{(B_K)}_\bar{g} \). Thus \( J^{(B_K)}_\bar{g} = \oplus \sum_{l \in L} J^g_l \) for each \( \bar{g} \in \bar{K} \). Conversely, since \( J^{(B_K)}_\bar{g} = \oplus \sum_{l \in L} J^g_l \) for each \( \bar{g} \in \bar{K} \), \( B_K = \oplus \sum_{\bar{g} \in \bar{K}} J^{(B_K)}_\bar{g} = \oplus \sum_{\bar{g} \in \bar{K}} J^{(B_K)}_{\bar{g}} \). Moreover, by Lemma 3.3, \( (B_K)^K = Z \), so \( \bar{K} \) is a \( Z \)-automorphism group of \( B_K \). Hence \( J^{(B_K)}_{\bar{g}} J^{(B_K)}_{\bar{g}^{-1}} = Z \) for each \( \bar{g} \in \bar{K} \). Thus \( B_K \) is a central Galois algebra with Galois group \( K|_{B_K} \equiv \bar{K} \) because \( B_K \) is an Azumaya \( Z \)-algebra by Theorem 3.1 (see [4, Theorem 1]).

Next, we characterize an Azumaya Galois extension \( B \) with Galois group \( K \).

\textbf{Theorem 4.2.} The following statements are equivalent:

\begin{enumerate}
\item\( B \) is an Azumaya Galois extension with Galois group \( K \);
\item\( Z = C \);
\item\( B = B_K B^K \);
\item\( B_K \) is a central Galois algebra over \( C \) with Galois group \( K|_{B_K} \equiv \bar{K} \).
\end{enumerate}

\textbf{Proof.} (1)\( \Rightarrow \) (2). Since \( B \) is an Azumaya Galois extension with Galois group \( K \), \( B^K \) is a \( C^K \)-Azumaya algebra. But, by Lemma 3.6, \( B^K \) is an Azumaya algebra over \( Z \), so \( Z = C^K \subset C \). Hence \( C \subset Z = C^K \subset C \). Thus \( Z = C \).

(2)\( \Rightarrow \) (3). Suppose that \( Z = C \). Then, by Theorem 3.1, \( B_K \) is an Azumaya algebra over \( C \). Hence by the commutator theorem for Azumaya algebras, \( B = B_K V_B(B_K) \) [3, Theorem 4.3, page 57]. But, by Lemma 3.6, \( B^K = V_B(B_K) \), so \( B = B_K B^K \).

(3)\( \Rightarrow \) (4). By hypothesis, \( B = B_K B^K \), so \( L = \{1\} \) where \( L \) is given in Lemma 3.2. By the proofs of Theorem 3.1 and Lemma 3.6, \( B_K \) and \( B^K \) are \( C \)-separable subalgebras of the Azumaya \( C \)-algebra \( B \) such that \( B = B_K B^K \), so \( B_K \) and \( B^K \) are Azumaya algebras over \( C \) [3, Theorem 4.4, page 58]. Thus \( C \) is the center of \( B_K \). Next, we claim that \( J_{\bar{g}} = J_{\bar{g}}^{(B_K)} \) for each \( \bar{g} \in \bar{K} \). In fact, it is clear that \( J_{\bar{g}} \subset J_{\bar{g}}^{(B_K)} \). Conversely, for each \( a \in J_{\bar{g}}^{(B_K)} \) and \( x \in B \) such that \( x = yz \) for some \( y \in B_K \) and \( z \in B^K \), noting that \( B^K = V_B(B_K) \), we have that \( ax = ayz = g(y)az = g(y)za = g(y)g(x)a = g(x)a \). Thus \( J_{\bar{g}}^{(B_K)} \subset J_{\bar{g}} \). This proves that \( J_{\bar{g}} = J_{\bar{g}}^{(B_K)} \) since \( L = \{1\} \) for each \( \bar{g} \in \bar{K} \). Hence, \( B_K \) is a central Galois algebra over \( C \) with Galois group \( K|_{B_K} \equiv K \) by Lemma 4.1.
(4)⇒(1). Since $B$ is a Galois algebra with Galois group $G$, $B$ is a Galois extension with Galois group $K$. By hypothesis, $B_K$ is a central Galois algebra over $C$ with Galois group $K|B_K = K$, so the center of $B_K$ is $C$, that is, $Z = C$. Hence $B^K$ is an Azumaya algebra over $C (= C^K)$ by Lemma 3.6. Thus $B$ is an Azumaya Galois extension with Galois group $K$. 

Theorem 4.2 generalizes the following result of Kanzaki [6, Proposition 3].

**Corollary 4.3.** If $f_g = \emptyset$ for each $g \notin K$, then $B$ is a central Galois algebra with Galois group $K$ and $C$ is a Galois algebra with Galois group $G/K$.

**Proof.** This is the case in Theorem 4.2 that $B = B_KB^K = B_K$ where $B^K = C$.

We conclude the present paper with two examples, one to illustrate the result in Theorem 4.2, and another to show that $Z \neq C$.

**Example 4.4.** Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = (A \otimes \mathbb{R}) \oplus A \oplus A \oplus A \oplus A$, and $G$ the group generated by the elements in \{g1, k1, j1, k-k, h1, h-j, h-k\} where $g_1$ is the identity of $G$ and for all $(a \otimes b, a_1, a_2, a_3, a_4) \in B$,

$$
k_i(a \otimes b, a_1, a_2, a_3, a_4) = (ia_1^{-1} \otimes b, ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}, ia_4i^{-1}),
$$

$$
k_j(a \otimes b, a_1, a_2, a_3, a_4) = (ja_1^{-1} \otimes b, ja_1j^{-1}, ja_2j^{-1}, ja_3j^{-1}, ja_4j^{-1}),
$$

$$
k_k(a \otimes b, a_1, a_2, a_3, a_4) = (kak^{-1} \otimes b, kak^{-1}, ka_2k^{-1}, ka_3k^{-1}, ka_4k^{-1}),
$$

$$
h_1(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes ibi^{-1}, a_2, a_1, a_4, a_3),
$$

$$
h_j(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes jb^{-1}, a_3, a_4, a_1, a_2),
$$

$$
h_k(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes kb^{-1}, a_4, a_3, a_2, a_1).
$$

Then,

1. we can check that $B$ is a Galois algebra over $B^G$ with Galois group $G$ where $B^G = \{(r_1 \otimes r_2, r_1, r_2) \mid r_1, r_2, r \in \mathbb{R}\} \subset C$, and $C = (\mathbb{R} \otimes \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of $B$;
2. $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{g_1, k_1, j_1, k-k\}$;
3. $J_1 = C$, $J_{k_1} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, $J_{k_2} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, $J_{k_3} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, so $B_K = (A \otimes \mathbb{R}) \oplus A \oplus A \oplus A \oplus A$. Hence $B_K$ has center $C$, that is $Z = C$, and $B_K$ is a central Galois algebra over $C$ with Galois group $K|B_K = K$;
4. $B^K = (\mathbb{R} \otimes \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and $B = B_KB^K$, that is, $B$ is an Azumaya Galois extension with Galois group $K$.

**Example 4.5.** Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = A \oplus A \oplus A$, $G = \{1, g_1, g_2, g_3\}$, and for all $(a_1, a_2, a_3) \in B$,

$$
g_i(a_1, a_2, a_3) = (ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}),
$$

$$
g_j(a_1, a_2, a_3) = (ja_1j^{-1}, ja_3j^{-1}, ja_2j^{-1}),
$$

$$
g_k(a_1, a_2, a_3) = (ka_1k^{-1}, ka_3k^{-1}, ka_2k^{-1}).
$$


Then,

1. $B$ is a Galois algebra over $B^G$ where $B^G = \{(r_1, r, r) \mid r_1, r \in \mathbb{R}\} \subset C$, and $C = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of $B$. The $G$-Galois system is $\{a_i; b_i \mid i = 1, 2, \ldots, 8\}$ where

\[
a_1 = (1, 0, 0), \quad a_2 = (i, 0, 0), \quad a_3 = (j, 0, 0), \quad a_4 = (k, 0, 0),
\]
\[
a_5 = (0, 1, 0), \quad a_6 = (0, j, 0), \quad a_7 = (0, 0, 1), \quad a_8 = (0, 0, k);
\]
\[
b_1 = \frac{1}{4} a_1, \quad b_2 = -\frac{1}{4} a_2, \quad b_3 = -\frac{1}{4} a_3, \quad b_4 = -\frac{1}{4} a_4, \quad (4.3)
\]
\[
b_5 = \frac{1}{2} a_5, \quad b_6 = -\frac{1}{2} a_6, \quad b_7 = \frac{1}{2} a_7, \quad b_8 = -\frac{1}{2} a_8,
\]

2. $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{1, g_1\}$ where $J_{g_1} = \mathbb{R}i \oplus \mathbb{R}i \oplus \mathbb{R}i$, so $B_K = \mathbb{R}[i] \oplus \mathbb{R}[i] \oplus \mathbb{R}[i]$ which is a commutative ring not equal to $C$, that is, $Z \neq C$.

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