CERTAIN INTEGRAL OPERATOR AND STRONGLY STARLIKE FUNCTIONS

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Let $S^*(\rho,\gamma)$ denote the class of strongly starlike functions of order $\rho$ and type $\gamma$ and let $C(\rho,\gamma)$ be the class of strongly convex functions of order $\rho$ and type $\gamma$. By making use of an integral operator defined by Jung et al. (1993), we introduce two novel families of strongly starlike functions $S^*_\alpha(\rho,\gamma)$ and $C^*_\beta(\rho,\gamma)$. Some properties of these classes are discussed.

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1. Introduction. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A$ is said to be starlike of order $\gamma$ if it satisfies

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in E)$$

(1.2)

for some $\gamma$ ($0 \leq \gamma < 1$). We denote by $S^*(\gamma)$ the subclass of $A$ consisting of functions which are starlike of order $\gamma$ in $E$. Also, a function $f(z)$ in $A$ is said to be convex of order $\gamma$ if it satisfies $zf'(z) \in S^*(\gamma)$, or

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in E)$$

(1.3)

for some $\gamma$ ($0 \leq \gamma < 1$). We denote by $C(\gamma)$ the subclass of $A$ consisting of all functions which are convex of order $\gamma$ in $E$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi \rho}{2} \quad (z \in E)$$

(1.4)

for some $\gamma$ ($0 \leq \gamma < 1$) and $\rho$ ($0 < \rho \leq 1$), then $f(z)$ is said to be strongly starlike of order $\rho$ and type $\gamma$ in $E$, and denoted by $f(z) \in S^*(\rho,\gamma)$. If $f(z) \in A$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right)\right| < \frac{\pi \rho}{2} \quad (z \in E)$$

(1.5)
for some \( \gamma \ (0 \leq \gamma < 1) \) and \( \rho \ (0 < \rho \leq 1) \), then we say that \( f(z) \) is strongly convex of order \( \rho \) and type \( \gamma \) in \( E \), and we denote by \( C(\rho, \gamma) \) the class of such functions. It is clear that \( f(z) \in A \) belongs to \( C(\rho, \gamma) \) if and only if \( zf'(z) \in S^*(\rho, \gamma) \). Also, we note that \( S^*(1, 1) = S^*(\gamma) \) and \( C(1, 1) = C(\gamma) \).

For \( c > -1 \) and \( f(z) \in A \), we recall the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) as

\[
L_c(f) = \frac{c + 1}{2^c} \int_0^z t^{c-1} f(t) dt.
\] (1.6)

The operator \( L_c(f) \) when \( c \in \mathbb{N} = \{1, 2, 3, \ldots\} \) was studied by Bernardi \[1\]. For \( c = 1 \), \( L_1(f) \) was investigated by Libera \[4\].

Recently, Jung et al. \[2\] introduced the following one-parameter family of integral operators:

\[
Q_{\alpha \beta}^\alpha f(z) = \left( \frac{\alpha + \beta}{\beta} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt \quad (\alpha > 0, \beta > -1, \ f \in A).
\] (1.7)

They showed that

\[
Q_{\alpha \beta}^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n) \Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n) \Gamma(\beta + 1)} a_n z^n,
\] (1.8)

where \( \Gamma(x) \) is the familiar Gamma function. Some properties of this operator have been studied (see \[2, 3\]). From (1.7) and (1.8), one can see that

\[
z(Q_{\alpha \beta}^{\alpha+1} f(z))' = (\alpha + \beta + 1) Q_{\alpha \beta}^\alpha f(z) - (\alpha + \beta) Q_{\alpha \beta}^{\alpha+1} f(z).
\] (1.9)

It should be remarked in passing that the operator \( Q_{\alpha \beta}^\alpha \) is related rather closely to the Beta or Euler transformation.

Using the operator \( Q_{\alpha \beta}^\alpha \), we now introduce the following classes:

\[
S_{\alpha \beta}^\alpha(\rho, \gamma) = \left\{ f(z) \in A : Q_{\alpha \beta}^\alpha f(z) \in S^*(\rho, \gamma), \frac{z(Q_{\alpha \beta}^\alpha f(z))'}{Q_{\alpha \beta}^\alpha f(z)} = \gamma \ \forall z \in E \right\},
\] (1.10)

\[
C_{\alpha \beta}^\alpha(\rho, \gamma) = \left\{ f(z) \in A : Q_{\alpha \beta}^\alpha f(z) \in C(\rho, \gamma), \frac{(z(Q_{\alpha \beta}^\alpha f(z))')'}{Q_{\alpha \beta}^\alpha f(z)} \neq \gamma \ \forall z \in E \right\}.
\]

It is obvious that \( f(z) \in C_{\alpha \beta}^\alpha(\rho, \gamma) \) if and only if \( zf'(z) \in S_{\alpha \beta}^\alpha(\rho, \gamma) \).

In this note, we investigate some properties of the classes \( S_{\alpha \beta}^\alpha(\rho, \gamma) \) and \( C_{\alpha \beta}^\alpha(\rho, \gamma) \). The basic tool for our investigation is the following lemma which is due to Nunokawa \[5\].

**Lemma 1.1.** Let a function \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) be analytic in \( E \) and \( p(z) \neq 0 \) \((z \in E)\). If there exists a point \( z_0 \in E \) such that

\[
|\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|), \quad |\arg p(z_0)| = \frac{\pi}{2} \rho \quad (0 < \rho \leq 1),
\]

(1.11)
then
\[ \frac{z_0p'(z_0)}{p(z_0)} = ik\rho, \tag{1.12} \]
where
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{\( \text{when arg} \ p(z_0) = \frac{\pi}{2} \rho \)}, \tag{1.13} \]
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{\( \text{when arg} \ p(z_0) = -\frac{\pi}{2} \rho \)}, \]
and \( p(z_0)^{1/\rho} = \pm ia \ (a > 0) \).

2. Main results. Our first inclusion theorem is stated as follows.

**Theorem 2.1.** The class \( S_\alpha^\rho(\rho, y) \subset S_{\alpha+1}^\rho(\rho, y) \) for \( \alpha > 0, \beta > -1, 0 \leq y < 1 \) and \( \alpha + \beta \geq -y \).

**Proof.** Let \( f(z) \in S_\alpha^\rho(\rho, y) \). Then we set
\[ \frac{z(Q_\beta^{\alpha+1}f(z))'}{Q_\beta^{\alpha+1}f(z)} = (1-y)p(z) + y, \tag{2.1} \]
where \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) is analytic in \( E \) and \( p(z) \neq 0 \) for all \( z \in E \). Using (1.9) and (2.1), we have
\[ (\alpha + \beta + 1) \frac{Q_\beta^\alpha f(z)}{Q_\beta^{\alpha+1}f(z)} = (\alpha + \beta + y) + (1-y)p(z). \tag{2.2} \]

Differentiating both sides of (2.2) logarithmically, it follows from (2.1) that
\[ \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - y' = (1-y)p(z) + \frac{(1-y)zp'(z)}{(\alpha + \beta + y) + (1-y)p(z)}. \tag{2.3} \]

Suppose that there exists a point \( z_0 \in E \) such that
\[ |\text{arg} \ p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|), \quad |\text{arg} \ p(z_0)| = \frac{\pi}{2} \rho. \tag{2.4} \]

Then, by applying Lemma 1.1, we can write that \( z_0p'(z_0)/p(z_0) = ik\rho \) and that \( (p(z_0))^{1/\rho} = \pm ia \ (a > 0) \).

Therefore, if \( \text{arg} \ p(z_0) = -(\pi/2)\rho \), then
\[ \frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - y' = (1-y)p(z_0) \left[ 1 + \frac{z_0p'(z_0)/p(z_0)}{(\alpha + \beta + y) + (1-y)p(z_0)} \right] \]
\[ = (1-y)a^\rho e^{-i\pi \rho/2} \left[ 1 + \frac{ik\rho}{(\alpha + \beta + y) + (1-y)a^\rho e^{-i\pi \rho/2}} \right]. \tag{2.5} \]
From (2.5) we have

\[
\arg\left\{ \frac{z_0(Q_\beta^{\alpha}f(z_0))'}{Q_\beta^{\alpha}f(z_0)} - y \right\} = -\frac{\pi}{2} \rho + \arg\left\{ 1 + \frac{ik\rho}{(\alpha + \beta + y) + (1 - y)a^\rho e^{-i\pi\rho/2}} \right\}
\]

\[
= -\frac{\pi}{2} \rho + \tan^{-1}\left\{ \frac{(\alpha + \beta + y)(1 - y)a^\rho \cos \frac{\pi \rho}{2}}{(\alpha + \beta + y)^2 + 2(\alpha + \beta + y)(1 - y)a^\rho \cos \frac{\pi \rho}{2} + (1 - y)^2 a^{2\rho} - k\rho(1 - y)a^\rho \sin \frac{\pi \rho}{2}} \right\}
\]

\[
\leq -\frac{\pi}{2} \rho,
\]

where \( k \leq -(1/2)(a + 1/a) \leq -1, \alpha + \beta \geq -y, \) which contradicts the condition \( f(z) \in S_\rho^{\alpha}(\rho, y). \)

Similarly, if \( \arg p(z_0) = (\pi/2)\rho, \) then we have

\[
\arg\left\{ \frac{z_0(Q_\beta^{\alpha}f(z_0))'}{Q_\beta^{\alpha}f(z_0)} - y \right\} \geq \frac{\pi}{2} \rho,
\]

which also contradicts the hypothesis that \( f(z) \in S_\rho^{\alpha}(\rho, y). \)

Thus the function \( p(z) \) has to satisfy \( |\arg p(z)| < (\pi/2)\rho \) \((z \in E), \) which leads us to the following:

\[
\left| \arg\left\{ \frac{z(Q_{\beta}^{\alpha+1}f(z))'}{Q_{\beta}^{\alpha+1}f(z)} - y \right\} \right| < \frac{\pi}{2} \rho \quad (z \in E).
\]

This evidently completes the proof of Theorem 2.1.

We next state the following theorem.

**Theorem 2.2.** The class \( C_\beta^{\alpha}(\rho, y) \subset C_\beta^{\alpha+1}(\rho, y) \) for \( \alpha > 0, \beta > -1, 0 \leq y < 1, \) and \( \alpha + \beta \geq -y. \)

**Proof.** By definition (1.10), we have

\[
f(z) \in C_\beta^{\alpha}(\rho, y) \iff Q_\beta^{\alpha}f(z) \in C(\rho, y) \iff z(Q_\beta^{\alpha}f(z))' \in S^*(\rho, y)
\]

\[
\iff Q_\beta^{\alpha}(zf'(z)) \in S^*(\rho, y) \iff zf'(z) \in S_\rho^{\alpha}(\rho, y)
\]

\[
\iff zf'(z) \in S_\rho^{\alpha+1}(\rho, y) \iff Q_\beta^{\alpha+1}(zf'(z)) \in S^*(\rho, y)
\]

\[
\iff z(Q_\beta^{\alpha+1}f(z))' \in S^*(\rho, y) \iff Q_\beta^{\alpha+1}f(z) \in C(\rho, y)
\]

\[
\iff f(z) \in C_\beta^{\alpha+1}(\rho, y).
\]

The following theorem involves the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) given by (1.6).
**Theorem 2.3.** Let \( c > -\gamma \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in A \) and \( z(Q_\beta^\alpha L_c f(z))'/Q_\beta^\alpha L_c f(z) \neq \gamma \) for all \( z \in E \), then \( f(z) \in S_\beta^\alpha (\rho, \gamma) \) implies that \( L_c(f) \in S_\beta^\alpha (\rho, \gamma) \).

**Proof.** Let \( f(z) \in S_\beta^\alpha (\rho, \gamma) \). Put

\[
\frac{z(Q_\beta^\alpha L_c f(z))'}{Q_\beta^\alpha L_c f(z)} = y + (1 - y)p(z),
\]

where \( p(z) \) is analytic in \( E \), \( p(0) = 1 \) and \( p(z) \neq 0 \) \((z \in E)\). From (1.6) we have

\[
z(Q_\beta^\alpha L_c f(z))' = (c + 1)Q_\beta^\alpha f(z) - c Q_\beta^\alpha L_c f(z).
\]

Using (2.10) and (2.11), we get

\[
(c + 1) \frac{Q_\beta^\alpha f(z)}{Q_\beta^\alpha L_c f(z)} = (c + y) + (1 - y)p(z).
\]

Differentiating both sides of (2.12) logarithmically, we obtain

\[
\frac{z(Q_\beta^\alpha L_c f(z))'}{Q_\beta^\alpha f(z)} - y = (1 - y)p(z) + \frac{(1 - y)zp'(z)}{(c + y) + (1 - y)p(z)}.
\]

Suppose that there exists a point \( z_0 \in E \) such that

\[
|\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|), \quad |\arg p(z_0)| = \frac{\pi}{2} \rho.
\]

Then, applying Lemma 1.1, we can write that \( z_0 p'(z_0)/p(z_0) = ik\rho \) and \((p(z_0))^{1/\rho} = \pm ia (a > 0)\).

If \( \arg p(z_0) = (\pi/2) \rho \), then

\[
z_0(Q_\beta^\alpha f(z_0))' - y = (1 - y)p(z_0) \left[ 1 + \frac{z_0 p'(z_0)/p(z_0)}{(c + y) + (1 - y)p(z_0)} \right] - y = (1 - y)a^\rho e^{i\pi\rho/2} \left[ 1 + \frac{ik\rho}{(c + y) + (1 - y)a^\rho e^{i\pi\rho/2}} \right].
\]

This shows that

\[
\arg \left\{ \frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - y \right\} = \frac{\pi}{2} \rho + \arg \left\{ 1 + \frac{ik\rho}{(c + y) + (1 - y)a^\rho e^{i\pi\rho/2}} \right\} = \frac{\pi}{2} \rho + \tan^{-1} \left\{ \left( k\rho \left( (c + y) + (1 - y)a^\rho \cos \frac{\pi\rho}{2} \right) \right) \times \left( (c + y)^2 + 2(c + y)(1 - y)a^\rho \cos \frac{\pi\rho}{2} + (1 - y)^2a^2 + k\rho (1 - y)a^\rho \sin \frac{\pi\rho}{2} \right)^{-1} \right\}
\]

\[
\geq \frac{\pi}{2} \rho,
\]

where \( k \geq (1/2)(a + 1/a) \geq 1 \), which contradicts the condition \( f(z) \in S_\beta^\alpha (\rho, \gamma) \).
Similarly, we can prove the case \( \arg p(z_0) = -\frac{\pi}{2}\rho \). Thus we conclude that the function \( p(z) \) has to satisfy \( |\arg p(z)| < \frac{\pi}{2}\rho \) for all \( z \in E \). This shows that

\[
\left| \arg \left\{ \frac{z(Q_\beta^\alpha L_c f(z))'}{Q_\beta^\alpha L_c f(z)} - y \right\} \right| < \frac{\pi}{2}\rho \quad (z \in E). \tag{2.17}
\]

The proof is complete.

**Theorem 2.4.** Let \( c > -\gamma \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in A \) and \( (z(Q_\beta^\alpha L_c f(z)))' / (Q_\beta^\alpha L_c f(z))' \neq \gamma \) for all \( z \in E \), then \( f(z) \in C_\beta^\alpha(\rho, \gamma) \) implies that \( L_c(f) \in C_\beta^\alpha(\rho, \gamma) \).

**Proof.** Using the same method as in Theorem 2.2 we have

\[
f(z) \in C_\beta^\alpha(\rho, \gamma) \iff zf'(z) \in S_\beta^\alpha(\rho, \gamma) \Rightarrow L_c(zf'(z)) \in S_\beta^\alpha(\rho, \gamma) \\
\iff z(L_c f(z))' \in S_\beta^\alpha(\rho, \gamma) \iff L_c f(z) \in C_\beta^\alpha(\rho, \gamma). \tag{2.18}
\]

**References**


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