ON $\beta$-DUAL OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

SUTHEP SUANTAI and WINATE SANHAN

Received 13 April 2001 and in revised form 10 October 2001

The $\beta$-dual of a vector-valued sequence space is defined and studied. We show that if an $X$-valued sequence space $E$ is a BK-space having AK property, then the dual space of $E$ and its $\beta$-dual are isometrically isomorphic. We also give characterizations of $\beta$-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

2000 Mathematics Subject Classification: 46A45.

1. Introduction. Let $(X, \| \cdot \|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let $\mathbb{N}$ be the set of all natural numbers, we write $x = (x_k)$ with $x_k$ in $X$ for all $k \in \mathbb{N}$. The $X$-valued sequence spaces of Maddox are defined as

$$c_0(X,p) = \left\{ x = (x_k) : \lim_{k \to \infty} \|x_k\|^p_k = 0 \right\};$$

$$c(X,p) = \left\{ x = (x_k) : \lim_{k \to \infty} \|x_k - a\|^p_k = 0 \text{ for some } a \in X \right\};$$

$$\ell_\infty(X,p) = \left\{ x = (x_k) : \sup_k \|x_k\|^p_k < \infty \right\};$$

$$\ell(X,p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^p_k < \infty \right\}.$$

(1.1)

When $X = \mathbb{K}$, the scalar field of $X$, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3, 4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p)$, $c(p)$, $\ell(p)$, and $\ell_\infty(p)$ and has given characterizations of $\beta$-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$, $1 < p < \infty$, is defined by

$$\ell_p[X] = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \text{ for each } f \in X' \right\}.$$

(1.2)

In this paper, the $\beta$-dual of a vector-valued sequence space is defined and studied and we give characterizations of $\beta$-dual of vector-valued sequence spaces of Maddox...
\(\ell(X, p), \ell_\infty(X, p), c_0(X, p),\) and \(c(X, p)\). Some results, obtained in this paper, are generalizations of some in \([1, 3]\).

2. Notation and definitions. Let \((X, \| \cdot \|)\) be a Banach space. Let \(W(X)\) and \(\Phi(X)\) denote the space of all sequences in \(X\) and the space of all finite sequences in \(X\), respectively. A sequence space in \(X\) is a linear subspace of \(W(X)\). Let \(E\) be an \(X\)-valued sequence space. For \(x \in E\) and \(k \in \mathbb{N}\) we write that \(x_k\) stand for the \(k\)th term of \(x\). For \(x \in X\) and \(k \in \mathbb{N}\), we let \(e^{(k)}(x)\) be the sequence \((0, 0, 0, \ldots, 0, x, 0, \ldots)\) with \(x\) in the \(k\)th position and let \(e(x)\) be the sequence \((x, x, x, \ldots)\). For a fixed scalar sequence \(u = (u_k)\), the sequence space \(E_u\) is defined as

\[
E_u = \{ x = (x_k) \in W(X) : (u_k x_k) \in E \}. \tag{2.1}
\]

An \(X\)-valued sequence space \(E\) is said to be normal if \((y_k) \in E\) whenever \(\|y_k\| \leq \|x_k\|\) for all \(k \in \mathbb{N}\) and \((x_k) \in E\). Suppose that the \(X\)-valued sequence space \(E\) is endowed with some linear topology \(\tau\). Then \(E\) is called a \(K\)-space if, for each \(k \in \mathbb{N}\), the \(k\)th coordinate mapping \(p_k : E \to X\), defined by \(p_k(x) = x_k\), is continuous on \(E\). In addition, if \((E, \tau)\) is a Fréchet (Banach) space, then \(E\) is called an FK-(BK)-space. Now, suppose that \(E\) contains \(\Phi(X)\), then \(E\) is said to have property AK if \(\sum_{k=1}^{\infty} e^{(k)}(x_k) \to x\) in \(E\) as \(n \to \infty\) for every \(x = (x_k) \in E\).

The spaces \(c_0(p)\) and \(c(p)\) are FK-spaces. In \(c_0(X, p)\), we consider the function \(g(x) = \sup_k \|x_k\|^{p_k/M}\), where \(M = \max\{1, \sup_k p_k\}\), as a paranorm on \(c_0(X, p)\), and it is known that \(c_0(X, p)\) is an FK-space having property AK under the paranorm \(g\) defined as above. In \(\ell(X, p)\), we consider it as a paranormed sequence space with the paranorm given by \(\|x_k\| = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/M}\). It is known that \(\ell(X, p)\) is an FK-space under the paranorm defined as above.

For an \(X\)-valued sequence space \(E\), define its Köthe dual with respect to the dual pair \((X, X')\) (see \([2]\)) as follows:

\[
E^\times|_{(X, X')} = \left\{ (f_k) \in X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \ \forall x = (x_k) \in E \right\}. \tag{2.2}
\]

In this paper, we denote \(E^\times|_{(X, X')}\) by \(E^\alpha\) and it is called the \(\alpha\)-dual of \(E\).

For a sequence space \(E\), the \(\beta\)-dual of \(E\) is defined by

\[
E^\beta = \left\{ (f_k) \in X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges} \ \forall (x_k) \in E \right\}. \tag{2.3}
\]

It is easy to see that \(E^\alpha \subseteq E^\beta\).

For the sake of completeness we introduce some further sequence spaces that will be considered as \(\beta\)-dual of the vector-valued sequence spaces of Maddox:

\[
M_0(X, p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} ||x_k||^{M^{-1}/p_k} < \infty \text{ for some } M \in \mathbb{N} \right\};
\]

\[
M_\infty(X, p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} ||x_k||^{1/p_k} < \infty \ \forall n \in \mathbb{N} \right\};
\]
\[ \ell_0(X, p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^p M^p k < \infty \text{ for some } M \in \mathbb{N}, \quad p_k > 1 \text{ } \forall k \in \mathbb{N} \right\}, \quad p_k > 1 \forall k \in \mathbb{N}; \]

\[ cs[X'] = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges } \forall x \in X \right\}. \tag{2.4} \]

When \( X = \mathbb{K} \), the scalar field of \( X \), the corresponding first two sequence spaces are written as \( M_0(p) \) and \( M_\infty(p) \), respectively. These two spaces were first introduced by Grosse-Erdmann [1].

3. Main results. We begin by giving some general properties of \( \beta \)-dual of vector-valued sequence spaces.

**Proposition 3.1.** Let \( X \) be a Banach space and let \( E, E_1, \) and \( E_2 \) be \( X \)-valued sequence spaces. Then

(i) \( E^\alpha \subseteq E^\beta \).

(ii) If \( E_1 \subseteq E_2 \), then \( E_2^\beta \subseteq E_1^\beta \).

(iii) If \( E = E_1 + E_2 \), then \( E^\beta = E_1^\beta \cap E_2^\beta \).

(iv) If \( E \) is normal, then \( E^\alpha = E^\beta \).

**Proof.** Assertions (i), (ii), and (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that \( E^\beta \subseteq E^\alpha \). Let \( (f_k) \in E^\beta \) and \( x = (x_k) \in E \). Then \( \sum_{k=1}^{\infty} f_k(x_k) \) converges. Choose a scalar sequence \( (t_k) \) with \( |t_k| = 1 \) and \( f_k(t_k x_k) = |f_k(x_k)| \) for all \( k \in \mathbb{N} \). Since \( E \) is normal, \( (t_k x_k) \in E \). It follows that \( \sum_{k=1}^{\infty} |f_k(x_k)| \) converges, hence \( (f_k) \in E^\alpha \). \( \square \)

If \( E \) is a BK-space, we define a norm on \( E^\beta \) by the formula

\[ \| (f_k) \|_{E^\beta} = \sup \left\{ \left\| \sum_{k=1}^{\infty} f_k(x_k) \right\| : \sum_{k=1}^{\infty} \|x_k\| \leq 1 \right\}. \tag{3.1} \]

It is easy to show that \( \| \cdot \|_{E^\beta} \) is a norm on \( E^\beta \).

Next, we give a relationship between \( \beta \)-dual of a sequence space and its continuous dual. Indeed, we need a lemma.

**Lemma 3.2.** Let \( E \) be an \( X \)-valued sequence space which is an FK-space containing \( \Phi(X) \). Then for each \( k \in \mathbb{N} \), the mapping \( T_k : X \to E \), defined by \( T_k x = e^k(x) \), is continuous.

**Proof.** Let \( V = \{ e^k(x) : x \in X \} \). Then \( V \) is a closed subspace of \( E \), so it is an FK-space because \( E \) is an FK-space. Since \( E \) is a K-space, the coordinate mapping \( p_k : V \to X \) is continuous and bijective. It follows from the open mapping theorem that \( p_k \) is open, which implies that \( p_k^{-1} : X \to V \) is continuous. But since \( T_k = p_k^{-1} \), we thus obtain that \( T_k \) is continuous. \( \square \)

**Theorem 3.3.** If \( E \) is a BK-space having property AK, then \( E^\beta \) and \( E' \) are isometrically isomorphic.
Proof. We first show that for \( x = (x_k) \in E \) and \( f \in E' \),
\[
f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)). \tag{3.2}
\]
To show this, let \( x = (x_k) \in E \) and \( f \in E' \). Since \( E \) has property AK,
\[
x = \lim_{n \to \infty} \sum_{k=1}^{n} e^{(k)}(x_k). \tag{3.3}
\]
By the continuity of \( f \), it follows that
\[
f(x) = \lim_{n \to \infty} \sum_{k=1}^{n} f(e^k(x_k)) = \sum_{k=1}^{\infty} f(e^k(x_k)), \tag{3.4}
\]
so (3.2) is obtained. For each \( k \in \mathbb{N} \), let \( T_k : X \to E \) be defined as in Lemma 3.2. Since \( E \) is a BK-space, by Lemma 3.2, \( T_k \) is continuous. Hence \( f \circ T_k \in X' \) for all \( k \in \mathbb{N} \). It follows from (3.2) that
\[
f(x) = \sum_{k=1}^{\infty} (f \circ T_k)(x) \quad \forall x = (x_k) \in E. \tag{3.5}
\]
It implies, by (3.5), that \( (f \circ T_k)_{k=1}^{\infty} \in E^\beta \). Define \( \varphi : E' \to E^\beta \) by
\[
\varphi(f) = (f \circ T_k)_{k=1}^{\infty} \quad \forall f \in E'. \tag{3.6}
\]
It is easy to see that \( \varphi \) is linear. Now, we show that \( \varphi \) is onto. Let \( (f_k) \in E^\beta \). Define \( f : E \to K \), where \( K \) is the scalar field of \( X \), by
\[
f(x) = \sum_{k=1}^{\infty} f_k(x_k) \quad \forall x = (x_k) \in E. \tag{3.7}
\]
For each \( k \in \mathbb{N} \), let \( p_k \) be the \( k \)th coordinate mapping on \( E \). Then we have
\[
f(x) = \sum_{k=1}^{\infty} (f_k \circ p_k)(x) = \lim_{n \to \infty} \sum_{k=1}^{n} (f_k \circ p_k)(x). \tag{3.8}
\]
Since \( f_k \) and \( p_k \) are continuous linear, so is also continuous \( f \circ p_k \). It follows by Banach-Steinhaus theorem that \( f \in E' \) and we have by (3.7) that, for each \( k \in \mathbb{N} \) and each \( z \in X \), \( (f \circ T_k)(z) = f(e^k(z)) = f_k(z) \). Thus \( f \circ T_k = f_k \) for all \( k \in \mathbb{N} \), which implies that \( \varphi(f) = (f_k) \), hence \( \varphi \) is onto.

Finally, we show that \( \varphi \) is linear isometry. For \( f \in E' \), we have
\[
\|f\| = \sup_{\|(x_k)\| \leq 1} |f((x_k))| \\
= \sup_{\|(x_k)\| \leq 1} \left| \sum_{k=1}^{\infty} f(e^k(x_k)) \right| \quad \text{(by (3.2))} \\
= \sup_{\|(x_k)\| \leq 1} \left| \sum_{k=1}^{\infty} (f \circ T_k)(x_k) \right| \\
= \left\| (f \circ T_k)_{k=1}^{\infty} \right\|_{E^\beta} \\
= \|\varphi(f)\|_{E^\beta}. \tag{3.9}
\]
Hence $\varphi$ is isometry. Therefore, $\varphi : E' \to E^\beta$ is an isometrically isomorphism from $E'$ onto $E^\beta$. This completes the proof. \qed

We next give characterizations of $\beta$-dual of the sequence space $\ell(X, p)$ when $p_k > 1$ for all $k \in \mathbb{N}$.

**Theorem 3.4.** Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$. Then $\ell(X, p)^\beta = \ell_0(X', q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$.

**Proof.** Suppose that $(f_k) \in \ell_0(X, q)$. Then $\sum_{k=1}^{\infty} \|f_k\| q_k M^{-q_k} < \infty$ for some $M \in \mathbb{N}$.

Then for each $x = (x_k) \in \ell(X, p)$, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{\infty} \|f_k\| M^{1/p_k} M^{1/p_k} |x_k|,$$

$$\leq \sum_{k=1}^{\infty} (\|f_k\| q_k M^{-q_k/p_k} + M |x_k|^{p_k})$$

$$= \sum_{k=1}^{\infty} \|f_k\| q_k M^{-q_k/p_k} + M \sum_{k=1}^{\infty} |x_k|^{p_k} \quad (3.10)$$

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X, p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X, p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X, p)$. For each $x = (x_k) \in \ell(X, p)$, choose scalar sequence $(t_k)$ with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in \ell(X, p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in \ell(X, p). \quad (3.11)$$

We want to show that $(f_k) \in \ell_0(X', q)$, that is, $\sum_{k=1}^{\infty} \|f_k\| q_k M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\| q_k m^{-q_k} = \infty \quad \forall m \in \mathbb{N}. \quad (3.12)$$

It implies by (3.12) that for each $k \in \mathbb{N}$,

$$\sum_{i>k} \|f_i\| q_i m^{-q_i} = \infty \quad \forall m \in \mathbb{N}. \quad (3.13)$$

By (3.12), let $m_1 = 1$, then there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{k<k_1} \|f_k\| q_k m_1^{-q_k} > 1. \quad (3.14)$$
By (3.13), we can choose \( m_2 > m_1 \) and \( k_2 > k_1 \) with \( m_2 > 2^2 \) such that
\[
\sum_{k_1 < k < k_2} \|f_k\|^{q_k} m_2^{-q_k} > 1. \tag{3.15}
\]
Proceeding in this way, we can choose sequences of positive integers \( (k_i) \) and \( (m_i) \) with \( 1 = k_0 < k_1 < k_2 < \cdots \) and \( m_1 < m_2 < \cdots \), such that \( m_i \) is such that
\[
\sum_{k_{i-1} < k < k_i} \|f_k\|^{q_k} m_i^{-q_k} > 1. \tag{3.16}
\]
For each \( i \in \mathbb{N} \), choose \( x_k \) in \( X \) with \( \|x_k\| = 1 \) for all \( k \in \mathbb{N} \), \( k_i - 1 < k \leq k_i \) such that
\[
\sum_{k_{i-1} < k < k_i} \|f_k(x_k)\|^{q_k} m_i^{-q_k} > 1 \quad \forall i \in \mathbb{N}. \tag{3.17}
\]
Let \( a_i = \sum_{k_{i-1} < k < k_i} |f_k(x_k)|^{q_k} m_i^{-q_k} \). Put \( y = (y_k) \), \( y_k = a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k \) for all \( k \in \mathbb{N} \) with \( k_{i-1} < k \leq k_i \). By using the fact that \( p_k q_k = p_k + q_k \) and \( p_k (q_k - 1) = q_k \) for all \( k \in \mathbb{N} \), we have that for each \( i \in \mathbb{N} \),
\[
\sum_{k_{i-1} < k < k_i} \|y_k\|^{p_k} = \sum_{k_{i-1} < k < k_i} \left| a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k \right|^{p_k}
= \sum_{k_{i-1} < k < k_i} a_i^{-p_k} m_i^{-p_k q_k} |f_k(x_k)|^{q_k}
= \sum_{k_{i-1} < k < k_i} a_i^{-p_k} m_i^{-p_k} m_i^{-q_k} |f_k(x_k)|^{q_k}
\leq a_i^{-1} m_i^{-1} \sum_{k_{i-1} < k < k_i} m_i^{-q_k} |f_k(x_k)|^{q_k}
\leq a_i^{-1} m_i^{-1} a_i
= m_i^{-1}
< \frac{1}{2^i},
\]
so we have that \( \sum_{k=1}^{\infty} \|y_k\|^{p_k} \leq \sum_{i=1}^{\infty} 1/2^i < \infty \). Hence, \( y = (y_k) \in \ell(X,p) \). For each \( i \in \mathbb{N} \), we have
\[
\sum_{k_{i-1} < k < k_i} |f_k(y_k)| = \sum_{k_{i-1} < k < k_i} \left| f_k \left( a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k \right) \right|
= \sum_{k_{i-1} < k < k_i} a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k}
= a_i^{-1} \sum_{k_{i-1} < k < k_i} m_i^{-q_k} |f_k(x_k)|^{q_k}
= 1, \tag{3.19}
\]
so that \( \sum_{k=1}^{\infty} |f_k(y_k)| = \infty \), which contradicts (3.11). Hence \( (f_k) \in \ell_0(X',q) \). The proof is now complete.
The following theorem gives a characterization of \( \beta \)-dual of \( \ell(X,p) \) when \( p_k \leq 1 \) for all \( k \in \mathbb{N} \). To do this, the following lemma is needed.

**Lemma 3.5.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers. Then \( \ell_\infty(X,p) = \bigcup_{n=1}^\infty \ell(X,(n^{-1/p_k})) \).

**Proof.** Let \( x \in \ell_\infty(X,p) \), then there is some \( n \in \mathbb{N} \) with \( \|x_k\|_{p_k} \leq n \) for all \( k \in \mathbb{N} \). Hence \( \|x_k\|_{n^{-1/p_k}} \leq 1 \) for all \( k \in \mathbb{N} \), so that \( x \in \ell(X,(n^{-1/p_k})) \). On the other hand, if \( x \in \bigcup_{n=1}^\infty \ell(X,(n^{-1/p_k})) \), then there are some \( n \in \mathbb{N} \) and \( M > 1 \) such that \( \|x_k\|_{n^{-1/p_k}} \leq M \) for every \( k \in \mathbb{N} \). Then we have \( \|x_k\|_{p_k} \leq nM^{p_k} \leq nM^\alpha \) for all \( k \in \mathbb{N} \), where \( \alpha = \sup_k p_k \). Hence \( x \in \ell_\infty(X,p) \).

**Theorem 3.6.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers with \( p_k \leq 1 \) for all \( k \in \mathbb{N} \). Then \( \ell(X,p)^\beta = \ell_\infty(X',p) \).

**Proof.** If \( (f_k) \in \ell(X,p)^\beta \), then \( \sum_{k=1}^\infty f_k(x_k) \) converges for every \( x = (x_k) \in \ell(X,p) \), using the same proof as in Theorem 3.4, we have

\[
\sum_{k=1}^\infty |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell(X,p).
\]

If \( (f_k) \notin \ell_\infty(X',p) \), it follows by Lemma 3.5 that \( \sup_k \|f_k\|_{m_i^{-1/p_k}} = \infty \) for all \( m \in \mathbb{N} \). For each \( i \in \mathbb{N} \), choose sequences \( (m_i) \) and \( (k_i) \) of positive integers with \( m_1 < m_2 < \cdots \) and \( k_1 < k_2 < \cdots \) such that \( m_i > 2^i \) and \( \|f_k\|_{m_i^{-1/p_{k_i}}} > 1 \). Choose \( x_{k_i} \in X \) with \( \|x_{k_i}\| = 1 \) such that

\[
|f_{k_i}(x_{k_i})| > m_{i-1}^{-1/p_{k_i}}.
\]

Let \( y = (y_k) \), \( y_k = m^{-1/p_{k_i}}x_{k_i} \) if \( k = k_i \) for some \( i \), and 0 otherwise. Then \( \sum_{k=1}^\infty \|y_k\|_{p_k} = \sum_{i=1}^\infty 1/m_i < \sum_{i=1}^\infty 1/2^i = 1 \), so that \( (y_k) \in \ell(X,p) \) and

\[
\sum_{k=1}^\infty |f_k(y_k)| = \sum_{i=1}^\infty \left| f_{k_i} \left( m_i^{-1/p_{k_i}}x_{k_i} \right) \right| = \sum_{i=1}^\infty m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| = \infty \quad \text{(by (3.21))},
\]

and this is contradictory to (3.20), hence \( (f_k) \in \ell_\infty(X',p) \).

Conversely, assume that \( (f_k) \in \ell_\infty(X',p) \). By Lemma 3.5, there exists \( M \in \mathbb{N} \) such that \( \sup_k \|f_k\|_{M_i^{-1/p_k}} < \infty \), so there is a \( K > 0 \) such that

\[
\|f_k\| \leq KM_i^{1/p_k} \quad \forall k \in \mathbb{N}.
\]

Let \( x = (x_k) \in \ell(X,p) \). Then there is a \( k_0 \in \mathbb{N} \) such that \( M_i^{1/p_k}\|x_k\| \leq 1 \) for all \( k \geq k_0 \). By \( p_k \leq 1 \) for all \( k \in \mathbb{N} \), we have that, for all \( k \geq k_0 \),

\[
M^{1/p_k}\|x_k\| \leq (M^{1/p_k}\|x_k\|)^{p_k} = M \|x_k\|^{p_k}.
\]
Then
\[ \sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{k_0} ||f_k|| ||x_k|| + \sum_{k=k_0+1}^{\infty} ||f_k|| ||x_k|| \]
\[ \leq \sum_{k=1}^{k_0} ||f_k|| ||x_k|| + K \sum_{k=k_0+1}^{\infty} M^{1/p_k} ||x_k|| \quad \text{(by (3.23))} \]
\[ \leq \sum_{k=1}^{k_0} ||f_k|| ||x_k|| + KM \sum_{k=k_0+1}^{\infty} ||x_k||^{p_k} \quad \text{(by (3.24))} \]
\[ < \infty. \]

This implies that \( \sum_{k=1}^{\infty} f_k(x_k) \) converges, hence \((f_k) \in \ell(X, p)\).

**Theorem 3.7.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers. Then \( \ell_\infty(X, p)^{\beta} = M_\infty(X', p) \).

**Proof.** If \((f_k) \in M_\infty(X', p)\), then \( \sum_{k=1}^{\infty} ||f_k|| M^{1/p_k} < \infty \) for all \( m \in \mathbb{N} \), we have that for each \( x = (x_k) \in \ell_\infty(X, p) \), there is \( m_0 \in \mathbb{N} \) such that \( ||x_k|| \leq m_0^{1/p_k} \) for all \( k \in \mathbb{N} \), hence \( \sum_{k=1}^{\infty} ||f_k(x_k)|| \leq \sum_{k=1}^{\infty} ||f_k|| ||x_k|| \leq \sum_{k=1}^{\infty} ||f_k|| m_0^{1/p_k} < \infty \), which implies that \( \sum_{k=1}^{\infty} f_k(x_k) \) converges, so that \((f_k) \in \ell_\infty(X, p)^{\beta}\).

Conversely, assume that \((f_k) \in \ell_\infty(X, p)^{\beta}\), then \( \sum_{k=1}^{\infty} f_k(x_k) \) converges for all \( x = (x_k) \in \ell_\infty(X, p) \), by using the same proof as in Theorem 3.4, we have
\[ \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell_\infty(X, p). \quad (3.26) \]

If \((f_k) \notin M_\infty(X', p)\), then \( \sum_{k=1}^{\infty} ||f_k|| M^{1/p_k} = \infty \) for some \( M \in \mathbb{N} \). Then we can choose a sequence \((k_i)\) of positive integers with \( 0 = k_0 < k_1 < k_2 < \cdots \) such that
\[ \sum_{k_{i-1} < k \leq k_i} ||f_k|| M^{1/p_k} > i \quad \forall i \in \mathbb{N}. \quad (3.27) \]

And we choose \( x_k \) in \( X \) with \( ||x_k|| = 1 \) such that for all \( i \in \mathbb{N} \),
\[ \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| M^{1/p_k} > i. \quad (3.28) \]

Put \( y = (y_k), y_k = M^{1/p_k} x_k \). Clearly, \( y \in \ell_\infty(X, p) \) and
\[ \sum_{k=1}^{\infty} |f_k(y_k)| = \sum_{k=1}^{\infty} |f_k(x_k)| M^{1/p_k} > i \quad \forall i \in \mathbb{N}. \quad (3.29) \]

Hence \( \sum_{k=1}^{\infty} |f_k(y_k)| = \infty \), which contradicts (3.26). Hence \((f_k) \in M_\infty(X', p)\). The proof is now complete.

**Theorem 3.8.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers. Then \( c_0(X, p)^{\beta} = M_0(X', p) \).
**Proof.** Suppose \((f_k) \in M_0(X', p)\), then \(\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty\) for some \(M \in \mathbb{N}\). Let \(x = (x_k) \in c_0(X, p)\). Then there is a positive integer \(K_0\) such that \(\|x_k\|^{p_k} < 1/M\) for all \(k \geq K_0\), hence \(\|x_k\| < M^{-1/p_k}\) for all \(k \geq K_0\). Then we have

\[
\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty. \tag{3.30}
\]

It follows that \(\sum_{k=1}^{\infty} f_k(x_k)\) converges, so that \((f_k) \in c_0(X, p)^\beta\).

On the other hand, assume that \((f_k) \in c_0(X, p)^\beta\), then \(\sum_{k=1}^{\infty} f_k(x_k)\) converges for all \(x = (x_k) \in c_0(X, p)\). For each \(x = (x_k) \in c_0(X, p)\), choose scalar sequence \((t_k)\) with \(|t_k| = 1\) such that \(f_k(t_kx_k) = f_k(x_k)\) for all \(k \in \mathbb{N}\). Since \((t_kx_k) \in c_0(X, p)\), by our assumption, we have \(\sum_{k=1}^{\infty} f_k(t_kx_k)\) converges, so that

\[
\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in c_0(X, p). \tag{3.31}
\]

Now, suppose that \((f_k) \notin M_0(X', p)\). Then \(\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} = \infty\) for all \(m \in \mathbb{N}\). Choose \(m_1, k_1 \in \mathbb{N}\) such that

\[
\sum_{k=k_1}^{\infty} \|f_k\| M^{-1/p_k} > 1 \tag{3.32}
\]

and choose \(m_2 > m_1\) and \(k_2 > k_1\) such that

\[
\sum_{k_1 < k < k_2} \|f_k\| M^{-1/p_k} > 2. \tag{3.33}
\]

Proceeding in this way, we can choose \(m_1 < m_2 < \cdots\), and \(0 = k_1 < k_2 < \cdots\) such that

\[
\sum_{k_{i-1} < k < k_i} \|f_k\| M^{-1/p_k} > i. \tag{3.34}
\]

Take \(x_k\) in \(X\) with \(\|x_k\| = 1\) for all \(k, k_{i-1} < k \leq k_i\) such that

\[
\sum_{k_{i-1} < k < k_i} |f_k(x_k)| M^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.35}
\]

Put \(y = (y_k), y_k = m_i^{-1/p_k} x_k\) for \(k_{i-1} < k \leq k_i\), then \(y \in c_0(X, p)\) and

\[
\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k < k_i} |f_k(x_k)| M^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.36}
\]

Hence we have \(\sum_{k=1}^{\infty} |f_k(y_k)| = \infty\), which contradicts (3.31), therefore \((f_k) \in M_0(X', p)\).

This completes the proof. \(\square\)

**Theorem 3.9.** Let \(p = (p_k)\) be a bounded sequence of positive real numbers. Then \(c(X, p)^\beta = M_0(X', p) \cap c_s[X']\).

**Proof.** Since \(c(X, p) = c_0(X, p) + E\), where \(E = \{e(x) : x \in X\}\), it follows by Proposition 3.1(iii) and Theorem 3.8 that \(c(X, p)^\beta = M_0(X', p) \cap E^\beta\). It is obvious by definition that \(E^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x)\) converges for all \(x \in X\} = c_s[X']\). Hence we have the theorem. \(\square\)
Acknowledgment. The author would like to thank the Thailand Research Fund for the financial support.

References


SUTHEP SUANTAI: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND
E-mail address: malsuthe@science.cmu.ac.th

WINATE SANHAN: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND