EQUIVALENCE RESULTS FOR DISCRETE ABEL MEANS

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We present theorems showing when the discrete Abel mean and the Abel summability method are equivalent for bounded sequences and when two discrete Abel means are equivalent for bounded sequences.

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1. Introduction and notation. The well-known Abel summability method is a sequence-to-function transformation which is defined as follows: for a sequence \( s := \{s_n\} \) of complex numbers, define

\[
f(x) := (1 - x) \sum_{k=0}^{\infty} s_k x^k,
\]

for all \( x \) for which the series converges. If \( f(x) \) exists for each \( x \in (0,1) \) and \( \lim_{x \to 1^-} f(x) = L \), then the sequence \( s \) is Abel summable to \( L \). The discrete Abel mean is a sequence-to-sequence transformation given by the summability matrix \( A_{\lambda} \) whose \( nk \)th entry is

\[
A_{\lambda}[n,k] := \frac{1}{\lambda(n)} \left( 1 - \frac{1}{\lambda(n)} \right)^k, \quad n, k = 0, 1, 2, 3, \ldots,
\]

where \( \lambda := \{\lambda(n)\} \) is a strictly increasing sequence of real numbers such that \( \lambda(0) \geq 1 \) and \( \lambda(n) \to \infty \). Then the sequence \( s \) is \( A_{\lambda} \)-summable to \( L \) provided that

\[
\lim_{n \to \infty} (A_{\lambda}s)_n = \lim_{n \to \infty} \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{\lambda(n)} \right)^k = L.
\]

In [1], Armitage and Maddox proved inclusion and Tauberian theorems for the discrete Abel mean. In this paper, we expand upon the work of these authors by examining equivalence properties of the \( A_{\lambda} \) method for bounded sequences.

For a given sequence \( s \), define a sequence \( a \) by \( a_0 := s_0 \) and \( a_n := s_n - s_{n-1} \) for \( n \geq 1 \). Then, \( s_n = \sum_{k=0}^{n} a_k \) and for every \( n \),

\[
(A_{\lambda}s)_n = \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{\lambda(n)} \right)^k = \sum_{k=0}^{\infty} a_k \left( 1 - \frac{1}{\lambda(n)} \right)^k.
\]

Also, define the sequence \( t \) by

\[
t_n := \sum_{k=1}^{n} ka_k.
\]
A straightforward induction argument yields

\[ t_n = \sum_{k=0}^{n} (s_n - s_k). \] (1.6)

If \( B \) and \( C \) are two summability methods, then \( C \) includes \( B \), denoted \( B \subset C \), provided that every sequence which is \( B \)-summable is also \( C \)-summable to the same limit. If \( B \subset C \) and \( C \subset B \), then \( B \) and \( C \) are equivalent, denoted \( B \sim C \).

2. Equivalence results. For any sequence \( \lambda \), \( A_{\lambda} \) is clearly a regular (i.e., limit preserving) method. In [1], Armitage and Maddox proved the following inclusion results for the \( A_{\lambda} \) method.

**Theorem 2.1** (see [1]). Let \( E(\lambda) := \{\lambda(n) : n = 0, 1, 2, \ldots\} \) and \( E(\mu) := \{\mu(n) : n = 0, 1, 2, \ldots\} \). Then

1. \( A_{\lambda} \subset A_{\mu} \) if and only if \( E(\mu) \setminus E(\lambda) \) is a finite set;
2. \( A_{\mu} \sim A_{\lambda} \) if and only if the symmetric difference \( E(\lambda) \triangle E(\mu) \) is a finite set.

**Corollary 2.2** (see [1]). For every \( \lambda \), \( A_{\lambda} \) strictly includes the Abel method.

The main result of this section is that \( A_{\lambda} \) is equivalent to the Abel method for bounded sequences provided that \( \lambda(n + 1)/\lambda(n) \to 1 \). To show this we need the following two lemmas.

**Lemma 2.3** (see [1]). If \( \sum_{k=0}^{\infty} a_k x^k \) converges for all \( x \in (0, 1) \), then

\[ \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} t_k \Delta \left( \frac{x^k}{k} \right), \quad 0 < x < 1, \] (2.1)

where \( \Delta(x^k/k) = x^k/k - x^{k+1}/(k+1) \).

**Lemma 2.4.** If \( s \) is a bounded sequence, then \( t_n = O(n) \).

**Proof.** Let \( s \) be a bounded sequence. By (1.6),

\[ |t_n| = \left| \sum_{k=0}^{n} (s_n - s_k) \right| = \left| (n+1)s_n - \sum_{k=0}^{n} s_k \right| \leq (n+1)\|s\|_{\infty} + \sum_{k=0}^{n} |s_k| \leq (n+1)\|s\|_{\infty} + (n+1)\|s\|_{\infty} = O(n). \]

\[ \square \]

**Theorem 2.5.** If \( \lim_{n \to \infty} (\lambda(n+1)/\lambda(n)) = 1 \), then \( A_{\lambda} \) is equivalent to the Abel method for bounded sequences.
Proof. By Corollary 2.2, $A_\lambda$ includes the Abel method. So assume that
\[
\lim_{n \to \infty} \left( \frac{\lambda(n+1)}{\lambda(n)} \right) = 1, \tag{2.3}
\]
let $s$ be a bounded sequence, that is, $A_\lambda$-summable to $L$, and let $a$ be the sequence such that $s_n = \sum_{k=0}^n a_k$. Let $x_n := 1 - 1/\lambda(n)$. Then, for a given $x \in (x_0, 1)$, there exists an $n$ such that $x_n < x \leq x_{n+1}$. By (1.1) and (1.4),
\[
|f(x) - (A_\lambda s)_n| = \left| (1-x) \sum_{k=0}^\infty s_k x^k - \frac{1}{\lambda(n)} \sum_{k=0}^\infty s_k \left(1 - \frac{1}{\lambda(n)}\right)^k - \sum_{k=0}^\infty a_k x^k + \sum_{k=0}^\infty a_k x_n^k \right|. \tag{2.4}
\]

By Lemma 2.3, this becomes
\[
|f(x) - (A_\lambda s)_n| = \left| \sum_{k=1}^\infty t_k \Delta \left(\frac{x^k}{k}\right) - \sum_{k=1}^\infty t_k \Delta \left(\frac{x_n^k}{k}\right) \right|
\]
\[
= \left| \sum_{k=1}^\infty t_k \int_{x_n}^x t^{k-1} (1-t) \, dt \right| \tag{2.5}
\]
\[
\leq \sum_{k=1}^\infty |t_k| \int_{x_n}^{x_{n+1}} t^{k-1} (1-t) \, dt.
\]
By Lemma 2.4, there exists an $M > 0$ such that $|t_k| \leq kM$. Hence,
\[
|f(x) - (A_\lambda s)_n| \leq M \sum_{k=1}^\infty k \int_{x_n}^{x_{n+1}} t^{k-1} (1-t) \, dt
\]
\[
= M \int_{x_n}^{x_{n+1}} (1-t) \sum_{k=1}^\infty k t^{k-1} \, dt
\]
\[
= M \int_{x_n}^{x_{n+1}} \frac{1}{1-t} \, dt
\]
\[
= -M \left( \log (1-x_{n+1}) - \log (1-x_n) \right)
\]
\[
= -M \left( \log \left( \frac{1}{\lambda(n+1)} \right) - \log \left( \frac{1}{\lambda(n)} \right) \right)
\]
\[
= M \log \left( \frac{\lambda(n+1)}{\lambda(n)} \right)
\]
\[
= o(1). \tag{2.6}
\]

Since $s$ is $A_\lambda$-summable to $L$, we see that $\lim_{x \to 1^-} f(x) = L$. That is, $s$ is Abel summable to $L$, and hence, $A_\lambda$ is equivalent to the Abel method for bounded sequences. \hfill \Box

The next theorem presents an equivalence relationship between the discrete Abel means when $\lambda$ and $\mu$ are asymptotic.
**Theorem 2.6.** Let \( \lambda \) and \( \mu \) be strictly increasing sequences of real numbers such that \( \lambda(0) \geq 1, \mu(0) \geq 1, \lambda(n) \to \infty, \mu(n) \to \infty, \) and \( \lim_{n \to \infty} (\mu(n)/\lambda(n)) = 1. \) Then \( A_{\lambda} \) is equivalent to \( A_{\mu} \) for bounded sequences.

**Proof.** We proceed as in the proof of Theorem 2.5. Let \( s \) be a bounded sequence and let \( a \) be the sequence such that \( s_n = \sum_{k=0}^{n} a_k. \) Let \( M(n) := \max\{\lambda(n), \mu(n)\}, m(n) := \min\{\lambda(n), \mu(n)\}, x_n := 1 - 1/m(n), \) and \( y_n := 1 - 1/M(n). \) Then \( 0 \leq x_n \leq y_n < 1 \) and for a given \( n, \)

\[
| (A_{\mu}s)_n - (A_{\lambda}s)_n | = \left| \frac{1}{\mu(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{\mu(n)} \right)^k - \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{\lambda(n)} \right)^k \right|
\]

\[
= \left| \frac{1}{M(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{M(n)} \right)^k - \frac{1}{m(n)} \sum_{k=0}^{\infty} s_k \left( 1 - \frac{1}{m(n)} \right)^k \right| \quad (2.7)
\]

By Lemma 2.3,

\[
| (A_{\mu}s)_n - (A_{\lambda}s)_n | = \left| \sum_{k=1}^{\infty} t_k \Delta \left( \frac{y_n^k}{k} \right) - \sum_{k=1}^{\infty} t_k \Delta \left( \frac{x_n^k}{k} \right) \right|
\]

\[
= \left| \sum_{k=1}^{\infty} t_k \int_{x_n}^{y_n} t^{k-1} (1-t) \, dt \right| \quad (2.8)
\]

\[
\leq \sum_{k=1}^{\infty} |t_k| \int_{x_n}^{y_n} t^{k-1} (1-t) \, dt.
\]

By Lemma 2.4, there exists an \( M > 0 \) such that \( |t_k| \leq kM. \) Hence,

\[
| (A_{\mu}s)_n - (A_{\lambda}s)_n | \leq M \sum_{k=1}^{\infty} \int_{x_n}^{y_n} t^{k-1} (1-t) \, dt
\]

\[
= M \int_{x_n}^{y_n} (1-t) \sum_{k=1}^{\infty} kt^{k-1} \, dt
\]

\[
= M \int_{x_n}^{y_n} \frac{1}{1-t} \, dt
\]

\[
= -M \left( \log (1-y_n) - \log (1-x_n) \right)
\]

\[
= -M \left( \log \left( \frac{1}{M(n)} \right) - \log \left( \frac{1}{m(n)} \right) \right)
\]

\[
= M \log \left( \frac{M(n)}{m(n)} \right)
\]

\[
= o(1),
\]

since \( \lim_{n \to \infty} (M(n)/m(n)) = \lim_{n \to \infty} (\mu(n)/\lambda(n)) = 1. \) Hence, if \( s \) is \( A_{\lambda} \)-summable to \( L, \) then

\[
0 \leq |(A_{\mu}s)_n - L| \leq |(A_{\mu}s)_n - (A_{\lambda}s)_n| + |(A_{\lambda}s)_n - L| = o(1) + o(1) = o(1). \quad (2.10)
\]
Similarly, if \( s \) is \( A_\mu \)-summable to \( L \), then
\[
0 \leq |(A_\lambda s)_n - L| \leq |(A_\lambda s)_n - (A_\mu s)_n| + |(A_\mu s)_n - L| = o(1) + o(1) = o(1). \tag{2.11}
\]
Thus, \( A_\lambda \) and \( A_\mu \) are equivalent for bounded sequences.

To see that \( \lim_{n \to \infty} (\mu(n)/\lambda(n)) = 1 \) is not a necessary condition in Theorem 2.6, simply consider the sequences \( \lambda(n) := n^2 \) and \( \mu(n) := n^3 \). Then
\[
\lim_{n \to \infty} \frac{\lambda(n+1)}{\lambda(n)} = \lim_{n \to \infty} \frac{\mu(n+1)}{\mu(n)} = 1, \tag{2.12}
\]
and hence, by Theorem 2.5, \( A_\lambda, A_\mu \), and the Abel method are all equivalent for bounded sequences. However, \( \lambda \) and \( \mu \) are not asymptotic.

References


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