INTEGRABILITY AND $L^1$-CONVERGENCE OF REES-STANOJEVIĆ SUMS WITH GENERALIZED SEMICONVEX COEFFICIENTS

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Integrability and $L^1$-convergence of modified cosine sums introduced by Rees and Stanojević (1973) under a class of generalized semiconvex null coefficients are studied, using Cesaro means of integral order.

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1. Introduction. Let

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx.$$  (1.2)

The problem of $L^1$-convergence of the Fourier cosine series (1.1) has been settled for various special classes of coefficients. Young [6] found that $a_n \log n = o(1)$, $n \to \infty$ is a necessary and sufficient condition for cosine series with convex ($\Delta^2 a_n \geq 0$) coefficients, and Kolmogorov [5] extended this result to the cosine series with quasi-convex $(\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty)$ coefficients. Later, Garrett and Stanojević [3] using modified cosine sums (1.2), proved the following theorem.

**THEOREM 1.1.** Let $\{a_n\}$ be a null sequence of bounded variation. Then the sequence of modified cosine sums

$$g_n(x) = S_n(x) - a_{n+1} D_n(x),$$  (1.3)

where $S_n(x)$ are the partial sums of the cosine series (1.1) and $D_n(x)$ is the Dirichlet kernel, converges in $L^1$-norm to $g(x)$, the pointwise sum of the cosine series, if and only if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, independent of $n$, such that

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon, \quad \text{for every } n.$$  (1.4)

This result contains as a special case a number of classical and neo-classical results. In particular, in [3] the following corollary to Theorem 1.1 is proved.
**Theorem 1.2.** Let \( \{a_n\} \) be a null sequence of bounded variation satisfying condition (1.4). Then the cosine series is the Fourier series of its sum \( g(x) \) and \( \|S_n(g) - g\| = o(1) \), \( n \to \infty \) is equivalent to \( a_n \log n = o(1) \), \( n \to \infty \).

In [2] Garrett and Stanojević proved the following theorem.

**Theorem 1.3.** If \( \{a_n\} \) is a null quasi-convex sequence, then \( g_n(x) \) converges to \( g(x) \) in the \( L^1 \)-norm.

**Definition 1.4** (see [4]). A sequence \( \{a_n\} \) is said to be semiconvex if \( \{a_n\} \to 0 \) as \( n \to \infty \), and

\[
\sum_{n=1}^{\infty} n |\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0),
\]

where \( \Delta^2 a_n = \Delta a_n - \Delta a_{n+1}, \Delta a_n = a_n - a_{n+1} \).

It may be remarked here that every quasi-convex null sequence is semi-convex. We generalize semiconvexity of null sequences in the following way: a null sequence \( \{a_n\} \) is said to be generalized semiconvex, if

\[
\sum_{n=1}^{\infty} n^\alpha |\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n| < \infty, \quad \text{for } \alpha > 0 \quad (a_0 = 0).
\]

For \( \alpha = 1 \), this class reduces to the class defined in [4]. The object of this paper is to show that Theorem 1.3 of Garrett and Stanojević [2] holds good for cosine sums (1.2) with generalized semi-convex null coefficients.

**2. Notation and formulae.** In what follows, we use the following notions [7]:

\[
S_n^0 = S_n = a_0 + a_1 + \cdots + a_n;
\]

\[
S_n^k = S_{n-1}^k + S_{n-1}^{k-1} + \cdots + S_{n-1}^1 + S_n^1, \quad k = 1, 2, \ldots, n = 0, 1, 2, \ldots;
\]

\[
A_n^0 = 1, \quad A_n^k = A_n^{k-1} + A_{n-1}^{k-1} + \cdots + A_1^{k-1} + A_0^{k-1} \quad k = 1, 2, \ldots, n = 0, 1, 2, \ldots.
\]

The \( A_n^k \)'s are called the binomial coefficients and are given by the following relation:

\[
\sum_{k=0}^{\infty} A_n^k x^k = (1 - x)^{(n+\alpha)}.
\]

whereas \( S_n^k \)'s are given by

\[
\sum_{k=0}^{\infty} S_n^k x^k = (1 - x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^k,
\]

and

\[
A_n^\alpha = \sum_{v=0}^{n} A_v^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1},
\]

\[
A_n^\alpha = \left( \frac{n + \alpha}{n} \right)^\alpha \frac{n^\alpha}{\Gamma(\alpha + 1)} \quad (\alpha \neq -1, -2, \ldots).
\]

The Cesaro means \( T_n^\alpha \) of order \( \alpha \) is denoted by \( T_n^\alpha = S_n^\alpha / A_n^\alpha \).
Also for $0 < x \leq \pi$, let

$$
\tilde{D}_0(x) = -\frac{1}{2} \cot \frac{x}{2},
$$

$$
\tilde{S}_n(x) = \tilde{D}_0(x) + \sin x + \sin 2x + \cdots + \sin nx,
$$

$$
\tilde{S}^1_n(x) = \tilde{S}_0(x) + \tilde{S}_1(x) + \tilde{S}_2(x) + \cdots + \tilde{S}_n(x),
$$

$$
\tilde{S}^2_n(x) = \tilde{S}_0(x) + \tilde{S}^1_1(x) + \tilde{S}^1_2(x) + \cdots + \tilde{S}^1_n(x),
$$

$$
\vdots
$$

$$
\tilde{S}^k_n(x) = \tilde{S}^{k-1}_0(x) + \tilde{S}^{k-1}_1(x) + \tilde{S}^{k-1}_2(x) + \cdots + \tilde{S}^{k-1}_n(x).
$$

The conjugate Cesaro means $\tilde{T}_k^\alpha$ of order $\alpha$ is denoted by $\tilde{T}_k^\alpha = \tilde{S}_k^\alpha / A_k^\alpha$. We use the following lemma for the proof of our result.

**Lemma 2.1 (see [1]).** If $\alpha \geq 0$, $p \geq 0$,

$$
\epsilon_n = o(n^{-p}),
$$

$$
\sum_{n=0}^{\infty} A_n^{\alpha+p} |\Delta^{\alpha+1} \epsilon_n| < \infty,
$$

then

$$
\sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \epsilon_n| < \infty \text{ for } -1 \leq \lambda \leq \alpha,
$$

$$
A_n^{\lambda+p} \Delta^\lambda \epsilon_n \text{ is of bounded variation for } 0 \leq \lambda \leq \alpha \text{ and tends to zero as } n \to \infty.
$$

**3. Main result.** The main result of this paper is the following theorem.

**Theorem 3.1.** If $\{a_n\}$ is a generalized semiconvex null sequence, then $g_n(x)$ converges to $g(x)$ in $L^1$-metric if and only if $\lim_{n \to \infty} \Delta a_n \log n = o(1)$, as $n \to \infty$.

**Proof.** We have

$$
g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx
$$

$$
= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x)
$$

$$
= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \quad (a_0 = 0)
$$

$$
= \sum_{k=1}^{n-1} \left( a_{k-1} - a_{k+1} \right) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin (n+1)x}{2 \sin x} - a_{n+1} D_n(x),
$$

(3.1)
where
\[ D_n(x) = \frac{\sin nx + \sin(n + 1)x}{2 \sin x}, \]
\[ gn(x) = \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n + 1)x}{2 \sin x} - a_{n+1} \frac{\sin nx}{2 \sin x} - a_n \frac{\sin(n + 1)x}{2 \sin x} \]
\[ = \frac{1}{2 \sin x} \sum_{k=1}^{n} (\Delta a_{k-1} + \Delta a_k) \sin kx + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x}. \]

Applying Abel’s transformation, we have
\[ gn(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \sum_{v=1}^{k} \sin v x + (\Delta a_{n-1} + \Delta a_n) \sum_{v=1}^{n} \sin v x \]
\[ + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x} \]
\[ = \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) (\bar{S}_k^0(x) - \bar{S}_0(x)) + (\Delta a_{n-1} + \Delta a_n) (\bar{S}_n^0(x) - \bar{S}_0(x)) \right] \]
\[ + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x} \]
\[ = \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_k^0(x) - \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_0(x) \right] \]
\[ + \frac{1}{2 \sin x} [ (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) - (\Delta a_{n-1} + \Delta a_n) \bar{S}_0(x) ] + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x} \]
\[ = \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) (\tilde{S}_k^0(x)) - (\Delta a_{n-1} + \Delta a_n) \tilde{S}_n^0(x) + a_2 \tilde{S}_0(x) \right] \]
\[ + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x}. \]

(3.3)

If we use Abel’s transformation \( \alpha \) times, we have similarly,
\[ g_n(x) = \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-\alpha} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \tilde{S}_k^{\alpha-1}(x) + \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \tilde{S}_{n-k+1}^{\alpha-1}(x) \right] \]
\[ + \frac{1}{2 \sin x} \left[ \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \tilde{S}_{n-k+1}^{\alpha-1}(x) + a_2 \tilde{S}_0(x) \right] + \Delta a_n \frac{\sin(n + 1)x}{2 \sin x}. \]

(3.4)

Since \( \bar{S}_n(x) \) and \( \bar{T}_n(x) \) are uniformly bounded on every segment \([\epsilon, \pi - \epsilon], \epsilon > 0 \),
\[ g(x) = \lim_{n \to \infty} g_n(x) \]
\[ = \frac{1}{2 \sin x} \left[ \sum_{k=1}^{\infty} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \tilde{S}_k^{\alpha-1}(x) + a_2 \tilde{S}_0(x) \right]. \]

(3.5)
Thus
\[
g(x) - g_n(x) = \frac{1}{2\sin x} \left[ \sum_{k=n-\alpha+1}^{\infty} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \tilde{S}_k^{\alpha-1}(x) - \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \tilde{S}_{n-k+1}^k(x) \right] \\
- \frac{1}{2\sin x} \left[ \sum_{k=1}^{\alpha} \Delta^k a_{n-k+1} \tilde{S}_{n-k+1}^k(x) \right] - \Delta a_n \sin(n+1)x \frac{\sin(n+1)x}{2 \sin x},
\]

\[
\|g(x) - g_n(x)\| \leq C \left[ \sum_{k=n-\alpha+1}^{\infty} |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \right] \left[ \int_0^{\pi} |\tilde{S}_k^{\alpha-1}(x)| dx \right] \\
+ C \left[ \sum_{k=1}^{\alpha} |\Delta^k a_{n-k}| \int_0^{\pi} |\tilde{S}_{n-k}^k(x)| dx + \sum_{k=1}^{\alpha} |\Delta^k a_{n-k+1}| \int_0^{\pi} |\tilde{S}_{n-k+1}^k(x)| dx \right] \\
+ \int_0^{\pi} |\Delta a_n \sin(n+1)x| dx,
\]

\[
\|g(x) - g_n(x)\| \leq C \left[ \sum_{k=n-\alpha+1}^{\infty} A_\alpha^k |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \right] \left[ \int_0^{\pi} |\tilde{T}_k^{\alpha}(x)| dx \right] \\
+ C \left[ \sum_{k=1}^{\alpha} A_\alpha^k |\Delta^k a_{n-k}| \int_0^{\pi} |\tilde{T}_{n-k}^k(x)| dx \right] \\
+ C \left[ \sum_{k=1}^{\alpha} A_\alpha^k |\Delta^k a_{n-k+1}| \int_0^{\pi} |\tilde{T}_{n-k+1}^k(x)| dx \right] \\
+ \int_0^{\pi} |\Delta a_n \sin(n+1)x| dx.
\]

The first three terms of the above inequality are of \(o(1)\) by Lemma 2.1 and the hypothesis of Theorem 3.1.

Moreover, since
\[
\int_0^{\pi} \left| \frac{\sin(n+1)x}{2 \sin x} \right| dx \leq C \log n, \quad n \geq 2,
\]

therefore
\[
\int_0^{\pi} \left| \frac{\Delta a_n \sin(n+1)x}{2 \sin x} \right| dx \sim \Delta a_n \log n.
\]
It follows that \( \int_0^{\pi} |g(x) - g_n(x)| \, dx \to 0 \), if and only if \( \Delta a_n \log n \to o(1) \) as \( n \to \infty \). This completes the proof of the theorem. \( \square \)

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**REFERENCES**


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