THE UNION PROBLEM ON COMPLEX MANIFOLDS

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Let $\Omega$ be a relatively compact subdomain of a complex manifold, exhaustable by Stein open sets. We give a necessary and sufficient condition for $\Omega$ to be Stein, in terms of $L^2$-estimates for the $\bar{\partial}$-operator, equivalent to the condition of Markoe (1977) and Silva (1978).

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1. Introduction. As indicated in [7], from the beginning of the theory of Stein spaces, the following question has held great interest: is a complex space, which is exhaustable by a sequence $X_1 \subset X_2 \subset \cdots$ of Stein subspaces, itself Stein?

In [1], the following is proved: every domain in $\mathbb{C}^m$ which is exhaustable by a sequence of Stein domains $B_1 \subset B_2 \subset \cdots$ is itself Stein, and this is shown to hold more generally for unramified Riemann domain $\mathcal{B}$ over $\mathbb{C}^m$ in [6]. In [11], the following is proved: let $X$ be a reduced complex space and $X_1 \subset X_2 \subset \cdots$ be an exhaustion of $X$ by Stein domains, if every pair $(X_j, X_{j+1})$ is Runge then $X = UX_j$ is Stein. Recently, Markoe [9] and Silva [10] proved the following: let $X$ be reduced and $X_1 \subset X_2 \subset \cdots$ be an exhaustion of $X$ by Stein domains. Then $X$ is Stein if and only if $H^1(X, \mathcal{O}) = 0$ ($\mathcal{O}$ being the structure sheaf of $X$).

More recently the following has been proved in [12]: let $\Omega_1 \subset \Omega_2 \subset \cdots$ be a sequence of open Stein subsets of a Stein space $X$, $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, and $\dim H^1(\Omega, \mathcal{O}) < \infty$. Then $\Omega$ is Stein.

Fornæss [4] produced an example to show that if $X_1 \subset X_2 \subset \cdots$ is a sequence of Stein manifolds, the limit manifold $X = \bigcup X_j$, in which each $X_j$ is an open submanifold, need not be Stein. But it is known that if the limit manifold is itself an open submanifold of a Stein manifold then the limit manifold is necessarily Stein.

This led Fornæss and Narasimhan to pose the following problem [5]: let $X$ be a Stein space and $\Omega_1 \subset \Omega_2 \subset \cdots$ an increasing sequence of Stein open sets in $X$. Is $\bigcup \Omega_j$ Stein? As indicated above this is the case when $X$ is a Stein manifold, but this question remains open in the general case.

In this paper, we consider the case where $X$ is a general complex manifold and $\Omega_1 \subset \Omega_2 \subset \cdots$ an increasing sequence of open Stein manifolds in $X$ such that $\Omega = \bigcup \Omega_j$ is relatively compact in $X$. We give a condition for $\Omega$ to be Stein, equivalent to Markoe’s and Silva’s condition and involving $L^2$-estimates for the $\bar{\partial}$ operator.

2. Preliminaries. Let $X$ be an $n$-dimensional complex manifold with a $C^\infty$ Hermitian metric. The space $L^2_{(p,q)}(X)$ of square integrable differential forms of type $(p,q)$ on $X$
is a Hilbert space under the scalar product,

\[ (f, g) = \int_X f^* \ast \bar{g}, \]

(2.1)

where \( \ast \) is the Hodge \( \ast \)-operator associated with the metric and orientation of \( X \).

Let \( \Omega_1 \subset \Omega_2 \subset \cdots \) be an increasing sequence of Stein open sets in \( X \) such that their union \( \Omega = \bigcup_{j=1}^{\infty} \Omega_j \) is relatively compact in \( X \).

The following theorem is our main result.

**Theorem 2.1.** The union \( \Omega \) is Stein if and only if given an \( f \in L^2_{(p,q)}(\Omega) \), which is \( \delta \)-closed in the sense of distributions, there is a \( u \in L^2_{(p,q-1)}(\Omega) \) such that \( \delta u = f \) in the sense of distributions and

\[ \|u\|_{L^2_{(p,q-1)}(\Omega)} \leq K \|f\|_{L^2_{(p,q)}(\Omega)}, \quad q > 0, \]

(2.2)

where \( K \) depends on \( \Omega \).

Let \( U \) be a bounded open set in \( \mathbb{C}^n \), and \( \mathcal{O} \) the structure sheaf of \( \mathbb{C}^n \). A section \( f = (f_1, \ldots, f_p) \in \Gamma(U, \mathcal{O}^p) \), where \( p > 0 \) is an integer, is \( L^2 \)-bounded if

\[ \|f\|_{L^2(U)} = \|f_1\|_{L^2(U)} + \cdots + \|f_p\|_{L^2(U)} < \infty. \]

(2.3)

We then denote all sections of \( \mathcal{O}^p \) over \( U \) that are \( L^2 \)-bounded by \( \Gamma_2(U, \mathcal{O}^p) \).

For the definition of \( L^2 \)-bounded sections of coherent analytic sheaves, we require the coherent analytic sheaf \( \mathcal{F} \) to be defined on a simply connected polycylinder neighborhood \( V \) of the closure of \( U \). Then by [8, Theorem 5, Section F, Chapter VI], there is an \( \mathcal{O} \)-homographic in another simply connected polycylinder neighborhood \( V' \) of the closure of \( U \),

\[ \mathcal{O}^p \xrightarrow{\lambda} \mathcal{F} \rightarrow 0, \]

(2.4)

where \( p > 0 \) is some integer; and \( f \in \Gamma(U, \mathcal{F}) \) is \( L^2 \)-bounded if \( f \in \Gamma_2(U, \mathcal{F}) := \lambda(\Gamma_2(U, \mathcal{O}^p)) \). It can be shown that \( \Gamma_2(U, \mathcal{F}) \) is independent of \( \lambda \) and \( p \), so that \( \Gamma_2(U, \mathcal{F}) \) is well defined.

Now let \( \Omega \) be a relatively compact subdomain of an \( n \)-dimensional complex manifold \( X \). An open subset \( Y \) of \( \Omega \) is said to be admissible for the coherent analytic sheaf \( \mathcal{F} \) defined in the neighborhood of the closure of \( \Omega \) in \( X \), if \( Y \) is Stein. There is a coordinate neighborhood \( V \) in the closure of \( \Omega \) in \( X \), and \( \hat{Y} \) is contained in the neighborhood of \( \hat{\Omega} \) where \( \hat{\mathcal{F}} \) is defined as \( f \in \Gamma(V, \mathcal{F}) \) which is \( L^2 \)-bounded if

\[ f \in \Gamma_2(Y, \mathcal{F}) := \{ g \in \Gamma(Y, \mathcal{F}) : \eta_*(g) \in \Gamma_2(\eta(Y), \eta_*(\mathcal{F})) \}, \]

(2.5)

where \( \eta \) is the restriction of the biholomorphic map \( V \to V' \) to \( Y \), and \( \eta_*(\mathcal{F}) \) is the zero direct image of \( \mathcal{F} \) on \( Y \).

Let \( \Omega \) be as in **Theorem 2.1** (then clearly \( \Omega \) is locally Stein). Let \( \mathcal{F} \) be a coherent analytic sheaf in a neighborhood of the closure of \( \Omega \). Then it is clear that \( \Omega \) is a finite union, \( \Omega = \bigcup_{j=1}^{m} U_j \), where each \( U_j \) is admissible for \( \mathcal{F} \). If \( \mathcal{V} = \{U_j\}_{j \in I}, I = \{1, \ldots, m\}, \ldots \)
where the $U_j$'s are as above, we say that $\mathcal{V}$ is a finite admissible cover of $\Omega$ for $\mathcal{F}$ and we define the $L^2$ (alternate) $q$-cochains of $\mathcal{V}$ with values in $\mathcal{F}$ as those cochains,

$$
c = (c_\alpha) \in C^q(\mathcal{V}, \mathcal{F}) = \prod_{\alpha \in I^{q+1}} \Gamma(U_\alpha, \mathcal{F}),
$$

$$
U_\alpha = U_{i_0} \cap \cdots \cap U_{i_q}, \quad \alpha = (i_0, \ldots, i_q),
$$

which are alternate and satisfy $c_\alpha \in \Gamma_2(U_\alpha, \mathcal{F})$ for all $\alpha \in I^{q+1}$. We denote by $C^q_2(\mathcal{V}, \mathcal{F})$ the space of $L^2$-bounded cochains.

The coboundary operator,

$$
\delta : C^q(\mathcal{V}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{V}, \mathcal{F}),
$$

maps $C^q_2(\mathcal{V}, \mathcal{F})$ into $C^{q+1}_2(\mathcal{V}, \mathcal{F})$. If $Z^q_2(\mathcal{V}, \mathcal{F}) = \{c \in C^q_2(\mathcal{V}, \mathcal{F}) : \delta c = 0\}$ and $B^q_2(\mathcal{V}, \mathcal{F}) = \delta C^{q-1}_2(\mathcal{V}, \mathcal{F})$, then as usual $B^q_2(\mathcal{V}, \mathcal{F}) \subseteq Z^q_2(\mathcal{V}, \mathcal{F})$ and we define $H^q_2(\mathcal{V}, \mathcal{F}) := Z^q_2(\mathcal{V}, \mathcal{F}) / B^q_2(\mathcal{V}, \mathcal{F})$ and call it the $L^2$-bounded cohomology of $\mathcal{V}$ with values in $\mathcal{F}$. We then have the following theorem.

**Theorem 2.2.** For any $q > 0$, the natural map

$$
H^q_2(\mathcal{V}, \mathcal{F}) \rightarrow H^q(\Omega, \mathcal{F})
$$

is an isomorphism.

We use Theorem 2.2 as a pivot to prove Theorem 2.1, but the proof of Theorem 2.2 is not given here, since it is similar to that of [2, Theorem].

### 3. A triangle of isomorphisms.

Let $\Omega$ be as in Theorem 2.1. By the end of the section Theorem 2.1 will be proved. If $U \neq \emptyset$ is an open set in $\tilde{\Omega}$, then $\mathcal{B}_p^0(U)$ is the Hilbert space of holomorphic $p$-forms $h$ on $\Omega \cap U$ such that

$$
\|h\|_{L^2(p,0)(\Omega \cap U)} < \infty.
$$

(3.1)

If $V$ is open in $\tilde{\Omega}$ with $\emptyset \neq V \subset U$, the restriction map $\gamma_\Omega^V : \mathcal{B}_p^0(U) \rightarrow \mathcal{B}_p^0(V)$ is defined. Then $\mathcal{B}_p^0 = \{\mathcal{B}_p^0(U), \gamma_\Omega^V\}$ is the canonical presheaf of $L^2$-holomorphic $p$-forms on $\tilde{\Omega}$. The associated sheaf $\mathcal{B}_p^0$ is the sheaf of germs of $L^2$-holomorphic $p$-forms on $\tilde{\Omega}$. We then have the following lemma.

**Lemma 3.1.** Let $\mathcal{D}^p$ be the sheaf of germs of holomorphic $p$-forms on $X$, and $\mathcal{V}$ a finite admissible cover of $\Omega$ for $\mathcal{D}^p$. Then the following diagram is an isomorphism triangle of cohomology groups:

$$
\begin{array}{ccc}
H^q_2(\mathcal{V}, \mathcal{D}^p) & \rightarrow & H^q(\Omega, \mathcal{D}^p) \\
\downarrow & & \downarrow \\
H^q(\tilde{\Omega}, \mathcal{B}_p^0) & & \\
\end{array}
$$

(3.2)

for $q \geq 1$ and $p \geq 0$. 

Proof. From Theorem 2.2 and the fact that any finite cover of $$\tilde{\Omega}$$ has a refinement $$\mathcal{U} = \{V_j\}_{j \in J}$$ such that $$\mathcal{U}_\Omega = \{V_j \cap \Omega\}_{j \in J}$$ is a finite admissible cover of $$\Omega$$ for $$\mathcal{D}^p$$, the lemma follows.

Now, using Hörmander’s $$L^2$$-estimates locally we get the following lemma.

Lemma 3.2. The cohomology group $$H^q(\tilde{\Omega}, \mathbb{B}^p_2)$$ is isomorphic to the quotient space

$$\{g : g \in L^2_{(p,q)}(\Omega) \text{ and } \bar{\partial} g = 0\} / \{\bar{\partial} h : h \in L^2_{p,q-1}(\Omega) \text{ and } \bar{\partial} h \in L^2_{(p,q)}(\Omega)\},$$

where $$\Omega$$ is as in Theorem 2.1.

Also the following lemma is proved in [3].

Lemma 3.3. If $$\Omega \Subset X$$ is Stein, where $$X$$ is a complex manifold, then given $$f \in L^2_{(p,q)}(\Omega)$$ with $$\bar{\partial} f = 0$$, there is $$u \in L^2_{(p,q-1)}(\Omega)$$ such that

$$\bar{\partial} u = f, \quad \|u\|_{L^2_{(p,q-1)}(\Omega)} \leq K \|f\|_{L^2_{(p,q)}(\Omega)},$$

where $$K$$ depends on $$\Omega$$.

To finish with the proof of Theorem 2.1 we remark that $$\mathcal{D}^0 = \mathcal{O}$$ is the structure sheaf of $$X$$ (as in Theorem 2.1), therefore Theorem 2.1 follows from Lemmas 3.1, 3.2, and 3.3, and from Markoe’s and Silva’s condition.

References


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