We study the principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem:

$$-\Delta u(x) = \lambda g(x)u(x), \ x \in D; \ \frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \ x \in \partial D,$$

where $\Delta$ is the standard Laplace operator, $D$ is a bounded domain with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function which changes sign on $D$ and $\alpha \in \mathbb{R}$. We discuss the relation between $\alpha$ and the principal eigenvalues.

2000 Mathematics Subject Classification: 35J15, 35J25.

1. Introduction. We investigate the property of principal eigenvalues for the boundary value problem

$$-\Delta u(x) = \lambda g(x)u(x), \ x \in D,$$

$$\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \ x \in \partial D,$$

where $D$ is a bounded region in $\mathbb{R}^N$ with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function which changes sign on $D$ and $\alpha \in \mathbb{R}$.

Such problems have been studied in recent years since Fleming [4] studied the following associated nonlinear problems arising in the study of population genetics:

$$u_t(x,t) = \Delta u + \lambda g(x)f(u), \ x \in D,$$

where $f$ is some function of class $C^1$ such that $f(0) = 0 = f(1)$.

Fleming's results suggested that nontrivial steady-state solutions were bifurcating the trivial solutions $u \equiv 0$ and $u \equiv 1$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x) = \lambda g(x)u(x), \ x \in D.$$

The study of the linear ordinary differential equation case, however, goes back to Bocher [3]. Attention has been confined mainly to the cases of Dirichlet ($\alpha = \infty$) and Neumann boundary conditions.

In the case of Dirichlet boundary conditions, it is well known (see [5]) that there exists a double sequence of eigenvalues for (1.1)

$$\cdots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \cdots,$$
\(\lambda^+_1(\lambda^-_1)\) being the unique positive (negative) principal eigenvalue, that is, (1.1) has solution \(u(v)\) which is positive in \(D\). It is also well known that the case where \(0 < \alpha < \infty\) is similar to the Dirichlet case.

In the case of Neumann boundary conditions, \(0\) is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if \(\int_D g(x)dx < 0\) \((> 0)\); in the case where \(\int_D g(x)dx = 0\) there are no positive and no negative principal eigenvalues.

We show that the set of \(\lambda\)'s such that \(\lambda\) is a principal eigenvalue of (1.1) is a bounded set and its bounds are independent of \(\alpha\), and also the positive principal eigenvalue \(\lambda\) of (1.1) is strictly an increasing function of \(\alpha\).

Our analysis is based on a method used by Hess and Kato [5]. Consider, for fixed \(\lambda\), the eigenvalue problem

\[
-\Delta u(x) - \lambda g(x)u(x) = \mu u(x), \quad x \in D, \\
\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D.
\]

We denote the lowest eigenvalue of (1.5) by \(\mu(\alpha,\lambda)\). Let

\[
A_{\alpha,\lambda} = \left\{ \int_D |\nabla \phi|^2 dx + \alpha \int_{\partial D} \phi^2 ds_x - \lambda \int_D g \phi^2 dx : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}
\]

when \(\alpha \geq 0\), it is clear that \(A_{\alpha,\lambda}\) is bounded below. It is shown in [6], by using variational arguments, that \(\mu(\alpha,\lambda) = \inf A_{\alpha,\lambda}\) and that an eigenfunction corresponding to \(\mu(\alpha,\lambda)\) does not change sign on \(D\). Thus, clearly, \(\lambda\) is a principal eigenvalue of (1.1) if and only if \(\mu(\alpha,\lambda) = 0\).

When \(\alpha < 0\), the boundedness below of \(A_{\alpha,\lambda}\) is no longer obvious a priori, and it is shown by Afrouzi and Brown [2].

2. Boundedness and monotonicity of principal eigenvalues. The following theorem is proved in [1, Theorem 1.8].

**Theorem 2.1.** If

\[
\lambda_1 = \inf \left\{ \int_D \left[ |\nabla \phi|^2 + q \phi^2 \right] dx + \alpha \int_{\partial D} \phi^2 ds_x : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}, \quad (2.1)
\]

where \(q \in L^\infty(D)\), then there exists \(\phi_1 \in W^{1,2}(D), \int_D \phi_1^2 dx = 1\), such that

\[
\lambda_1 = \int_D \left[ |\nabla \phi_1|^2 + q \phi_1^2 \right] dx + \alpha \int_{\partial D} \phi_1^2 ds_x. \quad (2.2)
\]

Moreover, \(\lambda_1\) is the principal eigenvalue and \(\phi_1 > 0\) is a principal eigenfunction of

\[
-\Delta u(x) + q(x)u(x) = \lambda u(x), \quad x \in D, \\
\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D. \quad (2.3)
\]
It is obvious that $\lambda_1$ is the principal eigenvalue of (1.1) if and only if 0 is the principal eigenvalue of

$$
-\Delta u(x) - \lambda_1 g(x)u(x) = \mu u(x), \quad x \in D,
$$

$$
\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0, \quad x \in \partial D.
$$

(2.4)

Here we are ready to prove one of the main results of this section about the uniformly boundedness of principal eigenvalues of (1.1) with respect to $\alpha$.

**Theorem 2.2.** There exist $m < 0$ and $M > 0$ such that if $\lambda$ is a principal eigenvalue of (1.1), then $\lambda \in [m, M]$ and also $m, M$ are independent of $\alpha$.

**Proof.** Suppose that $\lambda_1$ is a principal eigenvalue of (1.1). Then 0 is a principal eigenvalue of (2.4) and so by Theorem 2.1, we have

$$
0 = \inf \left\{ \int_D |\nabla \phi|^2 + \alpha \int_{\partial D} \phi^2 ds_x - \lambda_1 \int_D g \phi^2 dx : \phi \in W^{1,2}(D), \int_D \phi^2 dx = 1 \right\}. \quad (2.5)
$$

Now, by considering test functions $\phi_1, \phi_2 \in C_0^\infty (D)$ such that $\int_D \phi_1^2 dx > 0$ also $\int_D \phi_2^2 dx = 1$ and $\int_D g \phi_2^2 dx < 0$ we have

$$
0 \leq \int_D |\nabla \phi_1|^2 + \alpha \int_{\partial D} \phi_1^2 ds_x - \lambda_1 \int_D g \phi_1^2 dx \quad (2.6)
$$

and also

$$
0 \leq \int_D |\nabla \phi_2|^2 + \alpha \int_{\partial D} \phi_2^2 ds_x - \lambda_1 \int_D g \phi_2^2 dx. \quad (2.7)
$$

Hence from (2.6) and (2.7) we obtain, respectively,

$$
\lambda_1 \leq \frac{\int_D |\nabla \phi_1|^2 dx}{\int_D g \phi_1^2 dx}, \quad \frac{\int_D |\nabla \phi_2|^2 dx}{\int_D g \phi_2^2 dx} \leq \lambda_1. \quad (2.8)
$$

So by assuming $M = \frac{\int_D |\nabla \phi_1|^2 dx}{\int_D g \phi_1^2 dx}$ and $m = \frac{\int_D |\nabla \phi_2|^2 dx}{\int_D g \phi_2^2 dx}$, we have obtained $\lambda \in [m, M]$, and also we see that $m, M$ are independent of $\alpha$. \hfill \Box

In the case $0 < \alpha < \infty$, it is known [1, Lemmas 1.18 and 1.19] that problem (1.1) has the unique positive (negative) principal eigenvalue, that is, $\lambda^+_1 (\lambda^-_1)$, such that if $u$ and $v$ are being eigenfunctions corresponding to $\lambda^+_1$ and $\lambda^-_1$, respectively, then $\int_D g u^2 dx > 0$ and $\int_D g v^2 dx < 0$. Also in the case $\alpha < 0$, the following theorem [2, Theorem 5] is proved.

**Theorem 2.3.** There exists $\alpha_0 \leq 0$ such that

(i) if $\alpha < \alpha_0$, then (1.1) does not have a principal eigenvalue;

(ii) if $\alpha = \alpha_0$, then (1.1) has a unique principal eigenvalue with the corresponding eigenfunction $u_0$ such that $\int_D g(x)u_0^2(x)dx = 0$;

(iii) if $\alpha > \alpha_0$, then (1.1) has exactly two principal eigenvalues $\lambda$ and $\mu, \lambda < \mu$; if $u_0$ and $v_0$ are eigenfunctions corresponding to $\lambda < \mu$, respectively, then $\int_D g(x)u_0^2(x)dx < 0$ and $\int_D g(x)v_0^2(x)dx > 0$;
(iv) $\alpha_0 = 0$ if and only if $\int_D g(x) \, dx = 0$.

Now we prove the monotonicity of principal eigenvalues of (1.1) with respect to $\alpha$.

**Theorem 2.4.** Suppose that $\lambda_1$ is a principal eigenvalue of

$$
-\Delta u(x) = \lambda g(x) u(x), \quad x \in D,
$$

such that the corresponding principal eigenvalue, say $u_1$, satisfies $\int_D g u_1^2 \, dx > 0$. If $\alpha_2 > \alpha_1$ and $\lambda_2$, $u_2$ are, respectively, principal eigenvalue and eigenfunction of

$$
-\Delta u(x) = \lambda g(x) u(x), \quad x \in D,
$$

such that $\int_D g u_2^2 \, dx > 0$, then $\lambda_2 < \lambda_1$.

**Proof.** Since $\lambda_1$ is a principal eigenvalue of (2.9), so 0 is a principal eigenvalue of

$$
-\Delta u(x) - \lambda_1 g(x) u(x) = \mu u(x), \quad x \in D,
$$

and so we have

$$
0 = \int_D |\nabla u_1|^2 \, dx + \alpha_1 \int_{\partial D} u_1^2 \, ds - \lambda_1 \int_D g u_1^2 \, dx
$$

and also

$$
0 = \inf \left\{ \int_D |\nabla u|^2 \, dx + \alpha_2 \int_{\partial D} u^2 \, ds - \lambda_2 \int_D g u^2 \, dx : u \in W^{1,2}(D), \int_D u^2 \, dx = 1 \right\}.
$$

If $\lambda_2 \geq \lambda_1$, then

$$
0 = \int_D |\nabla u_1|^2 \, dx + \alpha_1 \int_{\partial D} u_1^2 \, ds - \lambda_1 \int_D g u_1^2 \, dx
$$

$$
> \int_D |\nabla u_1|^2 \, dx + \alpha_2 \int_{\partial D} u_1^2 \, ds - \lambda_2 \int_D g u_1^2 \, dx
$$

$$
\geq 0
$$

which is impossible. Hence $\lambda_2 < \lambda_1$ and the proof is complete. \qed

**References**


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