WEAKLY COMPACTLY GENERATED FRECHET SPACES

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ABSTRACT. It is proved that a weakly compact generated Frechet space is Lindelöf in the weak topology. As a corollary it is proved that for a finite measure space every weakly measurable function into a weakly compactly generated Frechet space is weakly equivalent to a strongly measurable function.


KEYWORDS AND PHRASES. Lindelöf spaces, weakly measurable functions, strongly measurable functions, universally measurable functions, Frechet spaces, weakly compact sets.

1. INTRODUCTION.

If E is a weakly compactly generated Banach space then it is proved in [7] that E, with weak topology, is Lindelöf. (A topological space is said to be Lindelöf if its every open covering has a countable subcovering.) In this note we extend this result to the case when E is a weakly compactly generated
Frechet space. Also, some consequences are obtained. All locally convex spaces are taken over the field of real numbers. By a Frechet space we mean a Hausdorff, metrizable, complete locally convex space; we use the notations of [4] for locally convex spaces. \( E' \) will always denote the topological dual of a locally convex space \( E \). A locally convex space is said to be weakly compactly generated if there exists an increasing sequence of \( \sigma(E,E') \)-compact subsets of \( E \) whose union is dense in \( E \).

**Theorem 1.** Let \( E \) be a weakly compactly generated Frechet space. Then \( (E, \sigma(E,E')) \) is a Lindelöf space and \( E \) is a Borel subset of \( (E', \sigma(E',E')) \), \( E' \) being the bidual of \( E \).

**Proof.** Let \( \{V_n\} \) be a sequence of \( O \)-nbd. base having the properties:

1. Each \( V_n \) is absolutely convex and closed,
2. \( (n+1)V_{n+1} \subset V_n \), for every \( n \).

We take \( \{A_n\} \) for an increasing sequence of weakly compact, absolutely convex subsets of \( E \) such that \( \bigcup_{n=1}^{\infty} A_n = H \) is dense in \( E \). We identify \( (E, \sigma(E,E')) \) as a subspace of \( R^{E'} \), with product topology. \( R^{E'} \) is a subset of the compact Hausdorff space \( \overline{R}^{E'} \), where \( \overline{R} = [-\infty, \infty] \). For an \( x \in \overline{R}^{E'} \), \( y \in \overline{R}^{E'} \), \( x+y \in \overline{R}^{E'} \) has the natural meaning. For a compact set \( A \subset \overline{R}^{E'} \) and a compact set \( B \subset \overline{R}^{E'} \), \( A+B \) is compact. Thus \( A + \overline{V}_n \) is a compact subset of \( \overline{R}^{E'} \) for each \( k \) and \( n \), \( \overline{V}_n \) being the closure of \( V_n \) in \( \overline{R}^{E'} \). We claim that \( \bigcap_{n=1}^{\infty} (H+V_n) = E \). Since \( H \) is dense in \( E \) and \( V_n \) is a \( O \)-nbd., \( H+V_n \supseteq E \) for every \( n \) and so \( \bigcap_{n=1}^{\infty} (H+V_n) \supseteq E \). Conversely, take an \( x \in \bigcap_{n=1}^{\infty} (H+V_n) \). This means there exists a sequence \( \{h_n\} \subset H \) and a sequence \( \{z_n\} \) with \( z_n \in \overline{V}_n \) for each \( n \), such that \( x = h_n + z_n \) for each \( n \). Fix \( n_0 \in N \) and \( \varepsilon > 0 \). Choose an \( n_1 > \max(n_0, \frac{1}{\varepsilon}) \) and take an \( n > n_1 \). Since \( V_{n_0} \supseteq V_n \), \( |f(z_n)| \leq \frac{1}{n} < \frac{1}{n_1} < \varepsilon \), for every \( f \in \overline{V}_{n_0} \) the polar of \( V_{n_0} \) ([4]). Thus \( f(x-h_n) \to 0 \), uniformly for \( f \in \overline{V}_{n_0} \). From this it follows that \( \{h_n\} \) is Cauchy in \( E \) which is complete. If \( h_n \to y \) in \( E \) it
is easy to verify that, as elements of $E^{\prime\prime}$, $x=y$. This proves the claim.

Thus, in weak topology, $E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{n} (A_k + V_n)$ is analytic and so is Lindelöf ([6]). Also $(E^\prime, \sigma(E^\prime, E^\prime))$ can be considered as a subspace of $E^{\prime\prime}$. Since $(A_k + V_n)$ is compact in $E^{\prime\prime}$, $(A_k + V_n) \cap E^\prime$ is closed in $(E^\prime, \sigma(E^\prime, E^\prime))$ and so $(H + V_n) \cap E^\prime$ is Borel in $(E^\prime, \sigma(E^\prime, E^\prime))$. Since $E = \bigcap_{n=1}^{\infty} (H + V_n) \cap E^\prime$, it follows that $E$ is Borel in $(E^\prime, \sigma(E^\prime, E^\prime))$.

**Remark.** Similar results for Banach spaces are proved in [2, Cor. 3.2].

In the following result, some results and notations of ([3]) are used. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $E$ a Hausdorff locally convex space. A function $f : X \to E$ is called weakly measurable if $h \circ f$ is $\mu$-measurable for every $h \in E^\prime$. It is proved in ([2]) that if $f : X \to E$ is weakly measurable that the image measure $\mu : \mathcal{B} \to \mathbb{R}$, $\nu(B) = \mu(f^{-1}(B))$, is a Baire measure on $(E, \sigma(E^\prime, E^\prime))$, $\mathcal{B}$ being the class of all Baire subsets of $(E^\prime, \sigma(E^\prime, E^\prime))$ ([2], [8]). Two weakly measurable functions $f_i : X \to E$, $i = 1, 2$ are said to be weakly equivalent if $h \circ f_1 = h \circ f_2$ a.e. $[\mu]$, for every $h \in E^\prime$. If $E$ is Frechet then $f : X \to E$ is called strongly measurable if there exists a sequence $\{f_n\}$ of $\mathcal{A}$-simple functions, $f_n : X \to E$, such that $f_n \to f$, pointwise a.e. $[\mu]$.

**Corollary 2.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $E$ a weakly compactly generated Frechet space, and $f : X \to E$ a weakly measurable function. Then $f$ is weakly equivalent to a strongly measurable function.

**Proof.** By ([3], Cor. 5) it is enough to show that image Baire measure on $(E, \sigma(E^\prime, E^\prime))$ is tight (cf. [2]). Since $(E, \sigma(E^\prime, E^\prime))$ is Lindelöf, Baire measures are $\tau$-additive (normal in the terminology of [5],[8]). By ([5], Theorems 3.3, 3.4) every Frechet space is universally measurable and so every $\tau$-smooth measure is tight. This proves the result.

**Remark.** In case $E$ is a Banach space, this result is implicit in ([2], p. 88(4), Theorem 5.4); if in addition $f$ is bounded this is proved in ([1], p. 88).
REFERENCES


