ON OSCULATORY INTERPOLATION BY TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. A short and simple proof is given that osculatory interpolation by trigonometric polynomials is always possible.

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It is an elementary fact that if $2n+1$ points $\theta_j$ with

$$-\pi \leq \theta_0 < \theta_1 < \ldots < \theta_{2n} < \pi$$

(1)
are given, then there exists an element $f$ of $T_n$, the class of trigonometric polynomials of degree $\leq n$;

$$f(e^{i\theta}) = \sum_{k=-n}^{n} a_k e^{ik\theta},$$

so that $f(e^{i\theta_j}) = w_j$ for $j = 0, 1, \ldots, 2n$, where $w_0, w_1, \ldots, w_{2n}$ is any given sequence of complex numbers. A proof (in the real case) is given in Example 5 on page 38 of Davis' book [1], and on page 53, Problem 13 asks, somewhat enigmatically, "Discuss the possibility of osculatory trigonometric interpolation."

In this note, we give a simple proof of a theorem that answers this problem and a bit more.

**Theorem.** Given two sets of $n$ complex numbers, $w_1, \ldots, w_n$ and $w'_1, \ldots, w'_n$, there exists a trigonometric polynomial $f$ of the form (2), with $a_0 = 0$, so that $f(e^{i\theta_j}) = w_j$ and $f'(e^{i\theta_j}) = w'_j$ for $j = 1, 2, \ldots, n$.

**Remark.** The osculatory case is where all the $w_j = 0$. Our theorem amounts to letting the $\theta_j$ coalesce in pairs. Our proof depends on a trick that does not seem to cover more general kinds of coalescence, for which there is surely a corresponding result. The lemma we use to prove the theorem sheds a little light on the problem considered in [2-4] about the number of vanishing coefficients in the square of a polynomial.

**Proof of the Theorem.** Let $T_n^0$ be the subclass of $T_n$ where $a_0 = 0$, so that $T_n^0$ is a vector space of dimension $2n$, and consider the $2n$ linear functionals consisting of point evaluations of elements of $T_n^0$ at the $e^{i\theta_j}$ and also of point evaluations of their first derivative at the $e^{i\theta_j}$, $j = 1, \ldots, n$. By standard considerations of linear algebra it is enough to prove that these functionals are linearly independent, or equivalently, that if $f \in T_n^0$ and if $f(e^{i\theta_j}) = f'(e^{i\theta_j}) = 0$ for $j = 1, \ldots, n$, then $f \equiv 0$. Let us suppose we have such an $f$. 
Now writing $z = e^{i\theta}$ for $z$ on the unit circle $T = \{|z| = 1\}$, we have

$$z^n f(z) = P(z) = \sum_{k=0}^{2n} b_k z^k,$$

where $P(z)$ is an algebraic polynomial of degree $2n$ whose coefficient $b_n$ of degree $n$ satisfies $b_n = 0$. Since the roots of $P$ are at the distinct points $e^{i\theta_j}$ and are all double roots, we see that $P$ is the square of a polynomial $Q$ of degree $n$ with $n$ roots on the unit circle. The next lemma then settles the question.

**Lemma.** Let $Q$ be a polynomial of degree $n$ with $n$ roots on the unit circle. Then the middle coefficient $b_n$ of $Q^2(z)$ does not vanish.

**Proof.** Let

$$Q(z) = \prod_{j=1}^{n} (z-e^{i\theta_j})$$

so that

$$\frac{Q^2(z)}{z^n} = \prod_{j=1}^{n} (z-e^{i\theta_j}) \prod_{j=1}^{n} \left(\frac{1}{z} - e^{-i\theta_j}\right).$$

Hence, on $\{|z| = 1\}$, we have

$$|Q(z)|^2 = A \frac{Q^2(z)}{z^n},$$

(3)

where

$$A = \prod_{j=1}^{n} (-e^{-i\theta_j}).$$

Now integrate both sides of (3) around $T$ with respect to the measure $d\theta/2\pi = (dz)/(2\pi i z)$ to get
\[ A b_n = \int_T |Q(z)|^2 \frac{dz}{2\pi i z} > 0. \]

Since \( |A| = 1 \), we see that \( b_n \neq 0 \).

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