AN EXAMPLE OF A BLOCH FUNCTION

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(Received November 3, 1978)

ABSTRACT. A Bloch function is exhibited which has radial limits of modulus one almost everywhere but fails to belong to \( H^p \), for each \( 0 < p < \infty \).

KEY WORDS AND PHRASES. Bloch function.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A78

1. INTRODUCTION.

The purpose of this note is to give an example which seems to be useful in settling several questions about Bloch functions.

Let \( E \) be the subset of the complex plane \( \mathbb{C} \) consisting of the closed unit disc together with the Gaussian integers \( \mathbb{Z}^2 \). Let \( G \) be the complement of \( E \) in \( \mathbb{C} \). Let \( g : D \to G \) be the analytic universal covering map of \( G \) given by the uniformization theorem (\( D \) denotes the unit disc).

PROPOSITION. The function \( g \) is an unbounded Bloch function with the properties

(i) \( g \) has a radial limit \( g(e^{i\theta}) \) at almost every point \( e^{i\theta} \) of the unit circle.
(ii) the function \( g(e^{i\theta}) \) is of modulus one almost everywhere on the unit circle,

(iii) \( g \) is the reciprocal of a singular inner function, and so \( g \) does not belong to any \( H^p \) class.

Bloch functions on the unit disc may be defined as those analytic functions \( f \) on \( D \) for which the radii of the schlicht discs in the range of \( f \) are bounded above. The Bloch functions are somewhat analogous to functions in the disc algebra--Bloch functions can be characterized (see [1]) as those analytic functions which are uniformly continuous when \( D \) is given the hyperbolic metric and \( \mathcal{C} \) the Euclidean metric. Since Bloch functions may be characterized (see [1]) as those analytic functions \( f \) on \( D \) for which the quantity \( |f'(z)| (1 - |z|^2) \) is bounded for \( z \in D \), it follows that the modulus of a Bloch function grows rather slowly--at most as fast as \( \log(1/(1 - |z|)) \). Because functions in the disc algebra and bounded functions have good boundary behaviour, it is natural to ask about boundary values of Bloch functions--in particular about radial boundary values. (It is shown in [4] that a Bloch function has a radial limit at a point of the unit circle if and only if it has a non-tangential limit there.)

In [5], Pommerenke gave an example of a Bloch function with radial limits almost nowhere. The example given here is constructed in a similar way, but it contrasts with Pommerenke's in that it shows that Bloch functions which have radial limits almost everywhere need not be particularly well-behaved.

The example answers a question posed by Joseph Cima (private communication). He asked whether a Bloch function which has radial limits
almost everywhere and has the additional property that the boundary function belongs to $L^p$ need be in $H^p$. The function $g$ provides a negative answer to this question since $g(e^{i\theta}) \notin L^\infty$ while $g \in H^p$ for any $0 < p \leq \infty$. In fact $g$ does not belong to the class $N^+$ (see [2] p. 25) which contains $H^p$ for every $p$.

PROOF. It is evident that $g$ is an unbounded Bloch function. Also, to verify properties (i), (ii) and (iii), it is clearly sufficient to verify (iii).

To establish (iii), consider the analytic function $f = 1/g$ on $D$. The function $f$ is bounded (by 1) and is the universal covering map: $D \rightarrow D - K$, where $K$ is the countable set

$$\{0\} \cup \{1/(m+in) | m, n \in \mathbb{Z}, |m+in| > 1\}.$$ 

Being a bounded analytic function, $f$ has radial limits almost everywhere on the unit circle. It is easy to see from the properties of covering maps that these radial limits are either of modulus 1 or else belong to $K$. To complete the proof that $f$ is a singular inner function, it is only necessary to show that the radial limit $f(e^{i\theta})$ belongs to $K$ on a subset of the unit circle of measure zero.

But, for each $k \in K$ it is true that the set of $e^{i\theta}$ for which $f(e^{i\theta}) = k$ has measure zero (see [2] p. 17). Since $K$ is countable, it follows that the set of $e^{i\theta}$ for which $f(e^{i\theta})$ belongs to $K$ also has measure zero. The proof is now complete.

The example may also be viewed as elucidating the almost total lack of relationships between the class $B$ of Bloch functions on $D$ and the subclasses $H^p$ and $N^+$ of the Nevanlinna class $N$ (see [2]). The only
The containment which holds between $\mathcal{B}$ and the other classes is the relation $\mathcal{H} \subseteq \mathcal{B}$. It is known that $\mathcal{H}^p \not\subseteq \mathcal{B}$ for any $0 < p < \infty$ and that $\mathcal{B} \not\subset \mathcal{N}$. The example $g$ given above belongs to $\mathcal{B} \cap \mathcal{N}$ but not to $\mathcal{N}^+$. The fact $\mathcal{B} \not\subset \mathcal{N}$ is shown by the example of Pommerenke's [5] mentioned above.

Finally, the example given here can be modified to show that there is no $\delta > 0$ such that an analytic function $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) = 1$$

almost everywhere on the unit circle must have a disc of radius $\delta$ in its range. (Merely replace $\mathbb{Z}^2$ by $\delta \mathbb{Z}^2$ in the construction of $g$). This answers a question raised by J.S. Hwang. By contrast, he showed (see [3]) that a singular inner function (for example) must have a (Schlicht) disc of radius at least $2B/e$ in its range, where $B$ denotes Bloch's constant.

**REFERENCES**