GLOBAL ATTRACTIVITY IN A GENOTYPE SELECTION MODEL

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We obtain a sufficient condition for the global attractivity of the genotype selection model

\[ y_{n+1} = y_n e^{\beta_n (1 - 2y_{n-k})/(1 - y_n + y_n e^{\beta_n (1 - 2y_{n-k})})}, \quad n \in \mathbb{N}, \]

where \( k \in \mathbb{N} \) and \( \{\beta_n\} \) is a sequence of positive real numbers.

When \( k = 0 \) and \( \beta_n \equiv \beta \) for all \( n \in \mathbb{N} \), (1.1) was introduced by May [2, pages 513–560] as an example of a map generated by a simple model for frequency-dependent natural selection. The local stability of the equilibrium \( \bar{y} = 1/2 \) of (1.1) was investigated by May [2]. In [1] (see also [3]), Grove further investigated the stability of the equilibrium \( \bar{y} = 1/2 \) of (1.1) and proved that when \( \beta_n \equiv \beta \), the equilibrium \( \bar{y} = 1/2 \) of (1.1) is locally asymptotically stable if \( 0 < \beta < 4 \cos(k \pi/(2k + 1)) \) and is unstable if \( 0 < \beta < 4 \cos(k \pi/(2k + 1)) \). Furthermore, if

\[ 0 < \beta \leq \frac{2}{k}, \quad k \in \mathbb{N}(1). \]

Then this equilibrium is a global attractor of all solution \( \{y_n\} \) of (1.1) with initial conditions \( y_{-k}, y_{-k+1}, \ldots, y_0 \in (0,1) \).

On the basis of computer observations, the authors of [1] also observe that condition (1.2) is probably far from sharp when \( k \in \mathbb{N}(2) \). Therefore, it is highly desirable to improve condition (1.2).

The purpose of this paper is to obtain new sufficient conditions for the global attractivity of the equilibrium \( \bar{y} = 1/2 \) of (1.1). Our main result is the following theorem.

**Theorem 1.1.** Assume that \( \{\beta_n\} \) is a positive sequence which satisfies

\[ \sum_{i=n-k}^{n} \beta_i \leq 3 + \frac{1}{k+1}, \quad (1.3) \]
for all large \( n \), and

\[
\sum_{i=0}^{\infty} \beta_i = \infty. \tag{1.4}
\]

Then every solution \( \{y_n\} \) of (1.1) with initial conditions \( y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1) \) will tend to \( \bar{y} = 1/2 \).

**Corollary 1.2.** Assume that \( \beta_n \equiv \beta \) for all \( n \in \mathbb{N} \) and

\[
\beta \leq \frac{3}{k+1} + \frac{1}{(k+1)^2}. \tag{1.5}
\]

Then every solution \( \{y_n\} \) of (1.1) with initial conditions \( y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1) \) will tend to \( \bar{y} = 1/2 \).

It is easy to see that when \( k \in \mathbb{N}(2) \), (1.5) is an improvement on (1.2).

By a solution of (1.1), we mean a sequence \( \{y_n\} \) that is defined for \( n \in \mathbb{N}(-k) \) and that satisfies (1.1) for \( n \in \mathbb{N} \). If \( a_{-k}, a_{-k+1}, \ldots, a_0 \) are \( k + 1 \) given constants, then (1.1) has a unique solution satisfying the initial conditions

\[
x_i = a_i \quad \text{for} \quad i \in \mathbb{N}(-k, 0). \tag{1.6}
\]

For the sake of convenience, throughout, we use the convention

\[
\sum_{n=i}^{j} r_n \equiv 0, \quad \text{whenever} \quad j \leq i-1. \tag{1.7}
\]

**2. Proof of Theorem 1.1.** Let \( \{y_n\} \) be a solution of (1.1) with initial conditions \( y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1) \). Then clearly, \( y_n \in (0, 1) \) for all \( n \in \mathbb{N}(-k) \). By introducing the substitution

\[
x_n = \ln \frac{y_n}{1-y_n}, \quad n \in \mathbb{N}(-k), \tag{2.1}
\]

we obtain

\[
\Delta x_n + r_n f(x_{n-k}) = 0, \quad n \in \mathbb{N}, \tag{2.2}
\]

\[
x_{-k}, x_{-k+1}, \ldots, x_0 \in (-\infty, \infty), \tag{2.3}
\]

where

\[
\Delta x_n = x_{n+1} - x_n, \quad r_n = \frac{1}{2} \beta_n, \quad f(x) = 2 - \frac{4}{e^x + 1}. \tag{2.4}
\]

It is easy to see that

\[
f(0) = 0, \quad xf(x) > 0 \quad \forall x \in \mathbb{R}, \tag{2.5}
\]

\[
f'(x) = \frac{4e^x}{(e^x + 1)^2} \quad \forall x \in \mathbb{R}. \tag{2.6}
\]
Thus, \( f \) is increasing, we also have
\[
f'(x) < \frac{4e^x}{(2\sqrt{e^x})^2} = 1 \quad \text{for } x \neq 0,
\]
which implies that
\[
|f(x)| < |x| \quad \text{for } x \neq 0.
\]
Define \( h \) as follows
\[
h(x) = \max \{ f(x), -f(-x) \} \quad \text{for } x > 0.
\]
We have from (2.5), (2.8), and the increasing property of \( f \) that \( h(x) \) is increasing in \([0, \infty)\), and
\[
|f(x)| \leq h(|x|) < |x| \quad \text{for } x \neq 0.
\]
We will now prove that
\[
\lim_{n \to \infty} x_n = 0.
\]
There are two cases to consider.

**CASE 1.** The sequence \( \{x_n\} \) is eventually nonnegative or eventually nonpositive. We assume that \( \{x_n\} \) is eventually nonnegative, then there exists an integer \( n_0 \in \mathbb{N}(k) \) such that \( x_{n-k} \geq 0 \) for all \( n \in \mathbb{N}(n_0) \). By (2.2), we have \( \Delta x_n \leq 0 \) for all \( n \in \mathbb{N}(n_0) \) and there exists \( a \geq 0 \) such that
\[
\lim_{n \to \infty} x_n = a.
\]
If \( a > 0 \), by the increasing property of \( f \), it follows that
\[
\Delta x_n \leq -r_n f(a) \quad \forall n \in \mathbb{N}(n_0 + k).
\]
Summing (2.13) from \( n_0 + k \) to \( n - 1 \) and using (1.4), we have
\[
x_n - x_{n_0 + k} \leq -f(a) \sum_{i=n_0+k}^{n-1} r_i \to -\infty \quad \text{as } n \to \infty,
\]
which contradicts (2.12). The case when \( \{x_n\} \) is eventually nonpositive can be dealt with similarly.

**CASE 2.** The sequence \( \{x_n\} \) is oscillatory. By (1.3) and (2.4), then there exists an integer \( n^* \in \mathbb{N}(2k) \) such that
\[
\sum_{i=n-k}^{n} r_i \leq \alpha = \frac{3}{2} + \frac{1}{2(k + 1)}, \quad n \in \mathbb{N}(n^* - 2k),
\]
\[
x_{n^*-1}x_{n^*} \leq 0, \quad x_{n^*} \neq 0.
\]
By virtue of the choice of \( n^* \), there exists a real number \( \lambda \in [0, 1) \) such that
\[
x_{n^*-1} + \lambda(x_{n^*} - x_{n^*-1}) = 0.
\]
Let \( l \) be a positive constant such that

\[
\max_{n \in \mathbb{N}(n^*-2k-1,n^*-1)} |x_n| \leq l. \tag{2.18}
\]

By (2.2), (2.10), (2.18), and the increasing property of \( h \), we have

\[
|\Delta x_n| \leq r_n h(l), \quad n \in \mathbb{N}(n^*-1,n^*+k-1). \tag{2.19}
\]

Which, together with (2.17), implies that

\[
|x_n - k| = |x_n - x_{n^*-1} - \lambda(x_{n^*} - x_{n^*-1})| = - \sum_{j=n-k}^{n^*-2} \Delta x_j - \lambda \Delta x_{n^*-1}\]

\[
\leq h(l) \left( \sum_{j=n-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right), \quad n \in \mathbb{N}(n^*-1,n^*+k-1). \tag{2.20}
\]

In view of (2.2), (2.10), and (2.20), we obtain

\[
|\Delta x_n| \leq r_n h(l) \left( \sum_{j=n-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right), \quad n \in \mathbb{N}(n^*-1,n^*+k-1). \tag{2.21}
\]

Now we show that

\[
|x_n| \leq h(l) \quad \forall n \in \mathbb{N}(n^*,n^*+k). \tag{2.22}
\]

There are two possible cases to consider.

**Case 1.** Suppose that \( d = \sum_{i=n^*+k-1}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} \leq 1 \). By (2.15), (2.17), and (2.21) we have for \( n \in \mathbb{N}(n^*,n^*+k) \)

\[
|x_n| = |x_n - x_{n^*-1} - \lambda(x_{n^*} - x_{n^*-1})| = \left| \sum_{i=n^*}^{n^*+k-1} \Delta x_i + (1-\lambda) \Delta x_{n^*-1} \right|
\]

\[
\leq \sum_{i=n^*}^{n^*+k-1} r_i h(l) \left( \sum_{j=i-k}^{i-1} r_j + \lambda r_{n^*-1} \right) + (1-\lambda)r_{n^*-1} h(l) \left( \sum_{j=n^*-k-1}^{n^*-2} r_j + \lambda r_{n^*-1} \right)
\]

\[
= h(l) \sum_{i=n^*}^{n^*+k-1} r_i \left[ \sum_{j=i-k}^{i-1} r_j - \sum_{j=n^*}^{i-1} r_j - (1-\lambda)r_{n^*-1} \right]
\]

\[
+ h(l)(1-\lambda)r_{n^*-1} \left[ \sum_{j=n^*-k-1}^{n^*-1} r_j - (1-\lambda)r_{n^*-1} \right]
\]
\[
\leq h(l) \left[ \alpha d - \sum_{i=n^*}^{n^*+k-1} r_i \sum_{j=n^*}^{i} r_j - (1-\lambda)r_{n^*-1}d \right]
\]
\[
= h(l) \left[ \alpha d - \frac{1}{2} \left( \sum_{i=n^*}^{n^*+k-1} r_i \right)^2 - \frac{1}{2} \sum_{i=n^*}^{n^*+k-1} r_i^2 - (1-\lambda)r_{n^*-1}d \right]
\]
\[
= h(l) \left[ \alpha d - \frac{1}{2} d^2 - \frac{1}{2} \left( \sum_{i=n^*}^{n^*+k-1} r_i^2 + (1-\lambda)^2 r_{n^*-1}^2 \right) \right].
\]

(2.23)

Since
\[
\sum_{i=n^*}^{n^*+k-1} r_i^2 + (1-\lambda)^2 r_{n^*-1}^2 \geq \frac{1}{k+1} \left( \sum_{i=n^*}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} \right)^2 = \frac{d^2}{k+1}.
\]

(2.24)

We obtain
\[
|x_n| \leq h(l) \left[ \alpha d - \left( \frac{1}{2} + \frac{1}{2(k+1)} \right)d^2 \right]
\]
\[
\leq h(l) \left[ \alpha - \left( \frac{1}{2} + \frac{1}{2(k+1)} \right) \right]
\]
\[
= h(l).
\]

(2.25)

**CASE 2.** Suppose that \(d = \sum_{i=n^*}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} > 1\). In this case, there exists an integer \(m \in \mathbb{N}(n^*, n^*+k)\) such that
\[
\sum_{i=m}^{n^*+k-1} r_i \leq 1, \quad \sum_{i=m-1}^{n^*+k-1} r_i > 1.
\]

(2.26)

Therefore, there is an \(\eta \in (0,1)\) such that
\[
\sum_{i=m}^{n^*+k-1} r_i + (1-\eta)r_{m-1} = 1.
\]

(2.27)

By (2.15), (2.17), (2.19), and (2.21), we have for \(n \in \mathbb{N}(n^*, n^*+k)\)
\[
|x_n| = |x_n - x_{n^*+1} - \lambda \Delta x_{n^*+1}|
\]
\[
= \left| \sum_{i=n^*}^{n^*+k-1} \Delta x_i + (1-\lambda) \Delta x_{n^*+1} \right|
\]
\[
= \sum_{j=n^*}^{n^*+k-1} |\Delta x_j| + (1-\lambda) |\Delta x_{n^*+1}|
\]
\[
= (1 - \lambda) |\Delta x_{n^* - 1}| + \sum_{j=n^*}^{m-2} |\Delta x_j| + \eta |\Delta x_{m-1}| + (1 - \eta) |\Delta x_m| + \sum_{j=m}^{n^* + k - 1} |\Delta x_j| \\
\leq h(l) \left( (1 - \lambda) r_{n^* - 1} + \sum_{j=n^*}^{m-2} r_j + \eta r_{m-1} \right) + h(l) (1 - \eta) r_{m-1} \left( \sum_{j=m-1-k}^{n^* - 2} r_j + \lambda r_{n^* - 1} \right) \\
+ h(l) \sum_{j=m}^{n^* + k - 1} r_j \left( \sum_{i=j-k}^{n^* - 2} r_i + \lambda r_{n^* - 1} \right) \\
= h(l) \left( (1 - \lambda) r_{n^* - 1} + \sum_{j=n^*}^{m-1} r_j - (1 - \eta) r_{m-1} \right) \\
+ h(l) (1 - \eta) r_{m-1} \left[ \sum_{j=m-1-k}^{m-1} r_j - \sum_{j=n^*}^{m-1} r_j - (1 - \lambda) r_{n^* - 1} \right] \\
+ h(l) \sum_{j=m}^{n^* + k - 1} r_j \left[ \sum_{i=j-k}^{j} r_i - \sum_{i=m}^{m-1} r_i - \sum_{i=n^*}^{n^* - 1} r_i - (1 - \lambda) r_{n^* - 1} \right] \\
\leq h(l) \left[ \alpha - (1 - \eta) r_{m-1} - \sum_{j=m}^{n^* + k - 1} j \sum_{i=m}^{n^* - 1} r_i \right] \\
= h(l) \left[ \alpha - (1 - \eta) r_{m-1} - \frac{1}{2} \left( \sum_{j=m}^{n^* + k - 1} r_j \right)^2 - \frac{1}{2} \sum_{j=m}^{n^* + k - 1} r_j^2 \right] \\
= h(l) \left[ \alpha - (1 - \eta) r_{m-1} - \frac{1}{2} (1 - (1 - \eta) r_{m-1})^2 - \frac{1}{2} \sum_{j=m}^{n^* + k - 1} r_j^2 \right] \\
= h(l) \left[ \alpha - \frac{1}{2} - \frac{1}{2} \left( \sum_{j=m}^{n^* + k - 1} r_j^2 + (1 - \eta)^2 r_{m-1}^2 \right) \right]. \\
\right]
\]

Since
\[
\sum_{j=m}^{n^* + k - 1} r_j^2 + (1 - \eta)^2 r_{m-1}^2 \geq \frac{1}{n^* - m + k + 1} \left( \sum_{j=m}^{n^* + k - 1} r_j + (1 - \eta) r_{m-1} \right)^2 \geq \frac{1}{k + 1}. \\
\]

We obtain
\[
|\Delta x_n| \leq h(l) \left( \alpha - \frac{1}{2} - \frac{1}{2(k + 1)} \right) = h(l). \\
\]

Furthermore, we can prove that
\[
|\Delta x_n| \leq h(l) \quad \forall n \in \mathbb{N}(n^*). \\
\]
Assume, for the sake of contradiction, that (2.31) is not true. Then there exists \( m_1 \in \mathbb{N}(n^* + k + 1) \) such that \( |x_{m_1}| > h(l) \) and \( |x_n| \leq h(l) \) for \( n \in \mathbb{N}(n^*, m_1 - 1) \). Set

\[
m_2 = \max \{ n \in \mathbb{N}(n^*, m_1) : x_{n-1}x_n \leq 0, x_n \neq 0 \}. \tag{2.32}
\]

In case \( m_1 \leq m_2 + k \). From (2.10), we have

\[
\max_{n \in \mathbb{N}(m_2 - 2k - 1, m_2 - 1)} |x_n| \leq h(l) < l. \tag{2.33}
\]

By a similar method to the proof of (2.22), we obtain

\[
|x_n| \leq h(l) \quad \forall n \in \mathbb{N}(m_2, m_2 + k) \tag{2.34}
\]

which contradicts the definition of \( m_1 \). In case \( m_1 - 1 \geq m_2 + k \), it follows from the choice of \( m_1 \) and \( m_2 \) that

\[
x_n > 0 \quad \text{or} \quad x_n < 0 \quad \forall n \in \mathbb{N}(m_2, m_1). \tag{2.35}
\]

Assume that \( x_n > 0 \) for all \( n \in \mathbb{N}(m_2, m_1) \). (In case \( x_n < 0 \), the proof is similar.) From (2.2) we have

\[
\Delta x_n \leq 0 \quad \text{for} \quad n \in \mathbb{N}(m_1 - 1, m_1 + k) \tag{2.36}
\]

which implies that

\[
x_{m_1} \leq x_{m_1 - 1} \leq h(l). \tag{2.37}
\]

This contradicts the definition of \( m_1 \). Thus (2.31) holds.

From the argument above, we can establish a sequence \( \{n_i\} \) of positive integers with \( n_1 = n^* \), \( n_{i+1} - n_i > 2k \) such that

\[
x_{n_{i-1}}x_{n_i} \leq 0, \quad x_{n_i} \neq 0, \tag{2.38}
\]

and a sequence \( \{z_i\} \) with \( z_1 = l \), \( z_{i+1} = h(z_i) \) such that

\[
\max_{n \in \mathbb{N}(n_i - 2k - 1, n_i - 1)} |x_n| \leq z_i, \quad |x_n| \leq z_{i+1} \quad \forall n \in \mathbb{N}(n_i). \tag{2.39}
\]

By (2.10), we obtain

\[
\lim_{i \to \infty} z_i = 0 \tag{2.40}
\]

which, together with (2.39), implies that \( \lim_{n \to \infty} x_n = 0 \). The proof is complete. \( \square \)

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