CERTAIN CONVEX HARMONIC FUNCTIONS

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We define and investigate a family of complex-valued harmonic convex univalent functions related to uniformly convex analytic functions. We obtain coefficient bounds, extreme points, distortion theorems, convolution and convex combinations for this family.

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1. Introduction. A continuous complex-valued function \( f = u + iv \) defined in a simply connected complex domain \( \mathbb{D} \subset \mathbb{C} \) is said to be harmonic in \( \mathbb{D} \) if both \( u \) and \( v \) are real harmonic in \( \mathbb{D} \). Consider the functions \( U \) and \( V \) analytic in \( \mathbb{D} \) so that \( u = \Re U \) and \( v = \Im V \). Then the harmonic function \( f \) can be expressed by

\[
f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{D},
\]

where \( h = (U + V)/2 \) and \( g = (U - V)/2 \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). If the co-analytic part of \( f \) is identically zero then \( f \) reduces to the analytic case.

The mapping \( z \mapsto f(z) \) is sense-preserving and locally one-to-one in \( \mathbb{D} \) if and only if the Jacobian of \( f \) is positive (see [1]), that is, if and only if

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \mathbb{D}.
\]

(1.2)

Let \( \mathcal{H} \) denote the family of functions \( f = h + \bar{g} \) which are harmonic, sense-preserving, and univalent in the open unit disk \( \Delta = \{ z : |z| < 1 \} \) with \( h(0) = f(0) = f_z(0) - 1 = 0 \). Thus, we may write

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.
\]

(1.3)

Also let \( \overline{\mathcal{H}} \) denote the subclass of \( \mathcal{H} \) consisting of functions \( f = h + \bar{g} \) so that the functions \( h \) and \( g \) take the form

\[
h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1.
\]

(1.4)

Recently, Kanas and Wisniowska [5] (see also Kanas and Srivastava [4]), studied the class of \( k \)-uniformly convex analytic functions, denoted by \( k-\mathcal{W}\mathcal{Y} \), \( 0 \leq k < \infty \), so that \( h \in k-\mathcal{W}\mathcal{Y} \) if and only if

\[
\Re \left\{ 1 + (z - \zeta) \frac{h''(z)}{h'(z)} \right\} \geq 0, \quad |\zeta| \leq k, \quad z \in \Delta.
\]

(1.5)
For real $\phi$ we may let $\zeta = -kze^{i\phi}$. Then condition (1.5) can be written as
\[
\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{zh''(z)}{h''(z)} \right\} \geq 0. \tag{1.6}
\]

Now considering the harmonic functions $f = h + \bar{g}$ of the form (1.3) we define the family $\mathcal{HV}(k, \alpha)$, $0 \leq \alpha < 1$, so that $f = h + \bar{g} \in \mathcal{HV}(k, \alpha)$ if and only if
\[
\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{zh''(z) + 2zg'(z) + z^2g''(z)}{zh'(z) - zg'(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1. \tag{1.7}
\]

Finally, we let $\mathcal{HV}(k, \alpha) = \mathcal{HV}(k, \alpha) \cap \mathcal{F}$.

Notice that if $g \equiv 0$ and $\alpha = 0$ then the family $\mathcal{HV}(k, \alpha)$ defined by (1.7) reduces to the class $k$-uniformly convex analytic functions defined by (1.5). If we, further, let $k = 1$ in (1.5), we obtain the class of uniformly convex analytic functions defined by Goodman [2]. A geometric characterization of the general family $\mathcal{HV}(k, \alpha)$ is an open question.

In Section 2, we introduce sufficient coefficient bounds for functions to be in $\mathcal{HV}(k, \alpha)$ and show that these bounds are also necessary for functions in $\mathcal{HV}(k, \alpha)$. In Section 3, the inclusion relation between the classes $k$-uniformly convex and $\mathcal{HV}(k, \alpha)$ is examined. Extreme points and distortion bounds for $\mathcal{HV}(k, \alpha)$ are given in Section 4. Finally, in Section 5, we show that the class $\mathcal{HV}(k, \alpha)$ is closed under convolution and convex combinations.

Here we state a result due to Jahangiri [3], which we will use throughout this paper.

**Theorem 1.1.** Let $f = h + \bar{g}$ with $h$ and $g$ of the form (1.3). If
\[
\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \leq 1, \quad 0 \leq \alpha < 1, \tag{1.8}
\]
then $f$ is harmonic, sense-preserving, univalent in $\Delta$, and $f$ is convex harmonic of order $\alpha$ denoted by $\mathcal{HC}(\alpha)$. Condition (1.8) is also necessary if $f \in \mathcal{HC}(\alpha) = \mathcal{HC}(\alpha) \cap \mathcal{F}$.

**2. Coefficient bounds.** First we state and prove a sufficient coefficient bound for the class $\mathcal{HV}(k, \alpha)$.

**Theorem 2.1.** Let $f = h + \bar{g}$ be of the form (1.3). If $0 \leq k < \infty$, $0 \leq \alpha < 1$, and
\[
\sum_{n=2}^{\infty} \frac{n(n+nk-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+nk+k+\alpha)}{1-\alpha} |b_n| \leq 1, \tag{2.1}
\]
then $f$ is harmonic, sense-preserving, univalent in $\Delta$, and $f \in \mathcal{HV}(k, \alpha)$.

**Proof.** Since $n - \alpha \leq n + nk - k - \alpha$ and $n + \alpha \leq n + nk + k + \alpha$ for $0 \leq k < \infty$, it follows from Theorem 1.1 that $f \in \mathcal{HC}(\alpha)$ and hence $f$ is sense-preserving and convex univalent in $\Delta$. Now, we only need to show that if (2.1) holds then
\[
\Re \left\{ \frac{zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + 2ke^{i\phi})zg'(z) + (1 + ke^{i\phi})z^2g''(z)}{zh'(z) - zg'(z)} \right\} = \Re \frac{A(z)}{B(z)} \geq \alpha. \tag{2.2}
\]
Using the fact that $\Re(w) \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$ it suffices to show that

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0, \quad (2.3)$$

where $A(z) = zh'(z) + (1 + ke^{i\phi})z^2 h''(z) + (1 + 2ke^{i\phi})zg'(z) + (1 + ke^{i\phi})z^2g''(z)$ and $B(z) = zh'(z) - zg'(z)$. Substituting for $A(z)$ and $B(z)$ in (2.3), we obtain

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)|$$

\[
\begin{align*}
&= |(2 - \alpha)z + \sum_{n=2}^{\infty} n(n + 1 - \alpha + k(n-1)\alpha) a_n z^n \\
&\quad + \sum_{n=1}^{\infty} n(n - 1 + \alpha + k(n+1)\alpha) \tilde{b}_n z^n | \\
&\quad - | - \alpha z + \sum_{n=2}^{\infty} n(n - 1 - \alpha + k(n-1)\alpha) a_n z^n \\
&\quad + \sum_{n=1}^{\infty} n(n + 1 + \alpha + k(n+1)\alpha) \tilde{b}_n z^n | \\
&\geq (2 - \alpha)|z| - \sum_{n=2}^{\infty} n(nk + 1 + 1 - k - \alpha) |a_n| |z|^n \\
&\quad - \sum_{n=1}^{\infty} n(nk + 1 - k + \alpha) |b_n| |z|^n \\
&\quad - |z| - \sum_{n=2}^{\infty} n(nk + 1 - k - \alpha) |a_n| |z|^n \\
&\quad - \sum_{n=1}^{\infty} n(nk + 1 + k + \alpha) |b_n| |z|^n \\
&\geq 2(1 - \alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{n(nk - k - \alpha)}{1 - \alpha} |a_n| \\
&\quad - \sum_{n=1}^{\infty} \frac{n(nk + k + \alpha)}{1 - \alpha} |b_n| \right\} \geq 0, \quad \text{by (2.1).} \quad (2.4)
\end{align*}
\]

The harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1 - \alpha}{n(nk + n - k - \alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{n(nk + n + k + \alpha)} y_n z^n, \quad (2.5)$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (2.5) are in $\mathcal{H} \mathcal{E} \mathcal{Y}(k, \alpha)$ because

$$\sum_{n=2}^{\infty} \frac{n(nk - k - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(nk + k + \alpha)}{1 - \alpha} |b_n| = \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \quad (2.6)$$
Next we show that the bound (2.1) is also necessary for functions in $\overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$.

**Theorem 2.2.** Let $f = h + \tilde{g}$ with $h$ and $\tilde{g}$ of the form (1.4). Then $f \in \overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$ if and only if

$$
\sum_{n=2}^{\infty} \frac{n(n + nk - k - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n + nk + k + \alpha)}{1 - \alpha} |b_n| \leq 1. \quad (2.7)
$$

**Proof.** In view of Theorem 2.1, we only need to show that $f \notin \overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$ if condition (2.7) does not hold. We note that a necessary and sufficient condition for $f = h + \tilde{g}$ given by (1.4) to be in $\overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$ is that the coefficient condition (1.7) to be satisfied. Equivalently, we must have

$$
\Re \left( (1 - \alpha)zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + \alpha + 2ke^{i\phi})zg'(z) + (1 + ke^{i\phi})z^2g''(z) \right) \geq 0.
$$

Upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$
1 - \alpha - \left[ \sum_{n=2}^{\infty} n(nk + n - k - \alpha)|a_n| + \sum_{n=1}^{\infty} n(nk + n + k + \alpha)|b_n| \right] r^{n-1} \geq 0. \quad (2.9)
$$

If condition (2.7) does not hold then the numerator in (2.9) is negative for $r$ sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient (2.9) is negative. This contradicts the required condition for $f \in \overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$ and so the proof is complete. \qed

3. Inclusion relations. As mentioned earlier in the proof of Theorem 2.1, the functions in $\overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$ are convex harmonic in $\Delta$. In the following example we show that this inclusion is proper.

**Example 3.1.** Consider the harmonic functions

$$
f_n(z) = z - \frac{1}{2}z^2 - \frac{1}{2n^2}z^n, \quad z \in \Delta, \ n = 2,3,\ldots \quad (3.1)
$$

For $a_n \equiv 0$ and $b_n = -1/2n^2$, we observe that

$$
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| = \frac{1}{2} + n^2 \left( \frac{1}{2n^2} \right) = \frac{1}{2} + \frac{1}{2} = 1. \quad (3.2)
$$

Therefore, by Theorem 1.1, $f_n \in \overline{\mathcal{K}}(0)$.

On the other hand,

$$
\frac{2k + 1 + \alpha}{1 - \alpha} - \frac{1}{2} + \frac{n(nk + n + k + \alpha)}{1 - \alpha} - \frac{1}{2n} \left( \frac{2k + 1 + \alpha}{2(1 - \alpha)} + \frac{n(2k + 1 + \alpha)}{2n(1 - \alpha)} \right) > 1. \quad (3.3)
$$

Thus, by Theorem 2.2, $f \notin \overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha)$.

More generally, we can prove the following theorem.

**Theorem 3.2.** Let $0 \leq k < \infty$, $0 \leq \alpha < 1$, and $0 \leq \beta < 1$. If $k > \beta/(1 - \beta)$ then the proper inclusion relation $\overline{\mathcal{E}} \cap \mathcal{V}(k, \alpha) \subset \overline{\mathcal{K}}(\beta)$. 

PROOF. Let \( f \in \mathcal{K} \in \mathcal{V}(k, \alpha) \), then, by Theorem 2.2,

\[
\sum_{n=2}^{\infty} \frac{n(nk+n-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| \leq 1. \tag{3.4}
\]

Since \((n-\beta)/(1-\beta) < (nk+n-k-\alpha)/(1-\alpha)\) and \((n+\beta)/(1-\beta) < (nk+n+k+\alpha)/(1-\alpha)\), by Theorem 1.1, we conclude that \( f \in \mathcal{K} \in \mathcal{V}(\beta) \).

To show that the inclusion is proper, consider the harmonic functions

\[
f_n(z) = z - \frac{1-\beta}{2(1-\beta)} z^2 - \frac{1-\beta}{2n(n+\beta)} z^n, \quad z \in \Delta, \quad n = 2, 3, \ldots. \tag{3.5}
\]

By Theorem 1.1, \( f_n \in \mathcal{K} \in \mathcal{V}(\beta) \), because

\[
\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta} |b_n| = \frac{1+\beta}{1-\beta} \frac{1-\beta}{2(1+\beta)} + \frac{1+\beta}{1-\beta} \frac{1-\beta}{2n(n+\beta)} = 1. \tag{3.6}
\]

On the contrary, by Theorem 2.2, \( f_n \not\in \mathcal{K} \in \mathcal{V}(k, \alpha) \), because

\[
\sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| = \frac{1+\alpha+2k}{1-\alpha} \frac{1-\beta}{2(1+\beta)} + \frac{n(n+\alpha+(n+1)k)}{1-\alpha} \frac{1-\beta}{2n(n+\beta)} \]

\[
= \frac{1-\beta}{2(1-\alpha)} \left\{ \frac{1+\alpha+2k}{1+\beta} + \frac{n+\alpha+(n+1)k}{n+\beta} \right\}
\]

\[
> \frac{1-\beta}{2(1-\alpha)} \left\{ \frac{1+\alpha+2\beta/(1-\beta)}{1+\beta} + \frac{n+\alpha+(n+1)\beta/(1-\beta)}{n+\beta} \right\}
\]

\[
= \frac{1}{2(1-\alpha)} \left\{ 2 + \frac{\alpha(1-\beta)(n+1+2\beta)}{(1+\beta)(n+\beta)} \right\} \geq 1. \tag{3.7}
\]

\( \square \)

4. Extreme points and distortion bounds. Using definition (1.7), and according to the arguments given in [3], we obtain the following extreme points of the closed convex hulls of \( \mathcal{K} \in \mathcal{V}(k, \alpha) \) denoted by \( \overline{\mathcal{K}} \in \mathcal{V}(k, \alpha) \).

**Theorem 4.1.** Let \( f \) be the form of (1.4). Then \( f \in \overline{\mathcal{K}} \in \mathcal{V}(k, \alpha) \) if and only if \( f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \) where \( h_1(z) = z, \ h_n(z) = z - ((1-\alpha)/n(n + nk - k - \alpha)) z^n (n = 2, 3, \ldots), \ g_n(z) = z - ((1-\alpha)/n(n + nk + k + \alpha)) z^n (n = 1, 2, 3, \ldots), \)\n
\( \sum_{n=1}^{\infty} (X_n + Y_n) = 1, \ X_n \geq 0 \ and \ Y_n \geq 0. \) In particular, the extreme points of \( \mathcal{K} \in \mathcal{V}(k, \alpha) \) are \( \{ h_n \} \) and \( \{ g_n \} \).

Similarly, follows the distortion bounds for functions in \( \mathcal{K} \in \mathcal{V}(k, \alpha) \).

**Theorem 4.2.** If \( f \in \mathcal{K} \in \mathcal{V}(k, \alpha) \) then

\[
|f(z)| \leq (1 + |b_1|) r + \frac{1}{2} \left( \frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1,
\]

\[
|f(z)| \geq (1 - |b_1|) r - \frac{1}{2} \left( \frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1. \tag{4.1}
\]
If we let $r \to 1$ in the left-hand inequality of Theorem 4.2 and collect the like terms, we obtain the following theorem.

**Theorem 4.3.** If $f \in \overline{H}(k, \alpha)$ then $\{w : |w| < (3 + 2k - \alpha)/2(2 + k - \alpha) - 3(1 - \alpha)/2(2 + k - \alpha)\} \subset f(\Delta)$.

5. **Convolutions and convex combinations.** For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n - \sum_{n=1}^{\infty} |b_n| z^n$ and $F(z) = z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=1}^{\infty} B_n z^n$, we define the convolution of $f$ and $F$ as

$$ (f \ast F)(z) = f(z) \ast F(z) = z - \sum_{n=2}^{\infty} |a_n| A_n z^n - \sum_{n=1}^{\infty} |b_n| B_n z^n. \quad (5.1) $$

In the following theorem we examine the convolution properties of the class $\overline{H}(k, \alpha)$.

**Theorem 5.1.** For $0 \leq \alpha \leq \beta < 1$, let $f \in \overline{H}(k, \beta)$ and $F \in \overline{H}(k, \alpha)$ then

$$ f \ast F \in \overline{H}(k, \beta) \subset \overline{H}(k, \alpha). \quad (5.2) $$

**Proof.** Express the convolution of $f$ and $F$ as that given by (5.1) and note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now the theorem follows upon the application of the required condition (2.7).

The convex combination properties of the class $\overline{H}(k, \alpha)$ is given in the following theorem.

**Theorem 5.2.** The class $\overline{H}(k, \alpha)$ is closed under convex combinations.

**Proof.** For $i = 1, 2, \ldots$, suppose that $f_i \in \overline{H}(k, \alpha)$ where $f_i$ is given by $f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n - \sum_{n=1}^{\infty} |b_{i,n}| z^n$. For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combinations of $f_i$ may be written as

$$ \sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i,n}| \right) z^n - \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{i,n}| \right) z^n. \quad (5.3) $$

Now, the theorem follows by (2.7) upon noting that $\sum_{i=1}^{\infty} t_i = 1$.

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