PURITY OF THE IDEAL OF CONTINUOUS FUNCTIONS WITH PSEUDOCOMPACT SUPPORT

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Let $C_\Psi(X)$ be the ideal of functions with pseudocompact support and let $kX$ be the set of all points in $\nu X$ having compact neighborhoods. We show that $C_\Psi(X)$ is pure if and only if $\beta X - kX$ is a round subset of $\beta X$, $C_\Psi(X)$ is a projective $C(X)$-module if and only if $C_\Psi(X)$ is pure and $kX$ is paracompact. We also show that if $C_\Psi(X)$ is pure, then for each $f \in C_\Psi(X)$ the ideal $(f)$ is a projective (flat) $C(X)$-module if and only if $kX$ is basically disconnected ($F_\sigma$-space).

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1. Introduction. Let $X$ be a completely regular $T_1$-space, $\beta X$ the Stone-Čech compactification of $X$ and $\nu X$ the Hewitt realcompactification of $X$. Let $C(X)$ be the ring of all continuous real-valued functions defined on $X$. For each $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$, $\text{coz} f = X - Z(f)$, the support of $f = S(f) = \text{cl}_X \text{coz} f$, and $S(f^\nu) = \text{cl}_{\nu X} S(f)$, where $f^\nu$ is the extension of $f$ to $\nu X$, $S(f^\beta) = \text{cl}_{\beta X} S(f^*)$, where

$$f^*(x) = \begin{cases} 1, & f(x) \geq 1, \\ f(x), & -1 \leq f(x) \leq 1, \\ -1, & f(x) \leq -1, \end{cases}$$

and $f^\beta$ is its extension to $\beta X$. If $I$ is an ideal in $C(X)$, then $\text{coz} I = \bigcup_{f \in I} \text{coz} f$.

Let $C_\kappa(X)$, $C_\Psi(X)$, and $I(X)$ be the ideal of functions with compact support, pseudocompact support, and the intersection of all free maximal ideals of $C(X)$, respectively.

The space $X$ is called $\mu$-compact if $C_\kappa(X) = I(X)$, it is called $\Psi$-compact if $C_\kappa(X) = C_\Psi(X)$, and it is called $\eta$-compact if $C_\Psi(X) = I(X)$.

Let $\mu X$ be the smallest $\mu$-compact subspace of $\beta X$ containing $X$, $\Psi X$ the smallest $\Psi$-compact subspace of $\beta X$ containing $X$, and $\eta X$ the smallest $\eta$-compact subspace of $\beta X$ containing $X$.

The following diagram illustrates the relationships between these spaces:

\[ \begin{array}{ccc} \beta X & \downarrow & \mu X \\ \nu X & \downarrow & \Psi X \\ \Psi X & \downarrow & \eta X \\ X & \end{array} \]
For more information about these spaces the reader may consult [7].

For each subset \( A \subseteq \beta X \), let \( M^A = \{ f \in C(X) : A \subseteq \text{cl}_\beta Z(f) \} \) and \( O^A = \{ f \in C(X) : A \subseteq \text{Int}_\beta \text{cl}_\beta Z(f) \} \). It is well known that \( C_K(X) = O^X - X \) and \( C_{\Psi}(X) = O^X - \nu X = M^X - \nu X \). A subset \( A \) of \( \beta X \) is called a round subset of \( \beta X \) if \( O^A = M^A \), see [10].

A space \( X \) is called locally pseudocompact if every point of \( X \) has a pseudocompact neighborhood (nbhd), it is called basically disconnected if for each \( f \in C(X) \), \( S(f) \) is clopen in \( X \) and it is called an \( F' \)-space if for each \( f, g \in C(X) \) such that \( fg = 0 \), then \( S(f) \cap S(g) = \emptyset \).

An ideal \( I \) of \( C(X) \) is called pure if for each \( f \in I \), there exists \( g \in I \) such that \( f = fg \). It is clear that in this case \( g = 1 \) on \( S(f) \).

Purity attracted the attention of a lot of people working in ring and module theories. A large class of commutative rings can be classified through the pure ideals of the ring. Purity of some ideals in \( C(X) \) was studied by many authors. Kohls [8, Theorem 4.6] called it an ideal with every element having a relative identity. Brookshear [3, page 325] proved that if \( X \) is locally compact, then \( C_K(X) \) is pure, Brookshear [3] and De Marco [4] studied purity and projectivity, Natsheh and Al-Ezeh [11, Theorem 2.4] characterized pure ideals in \( C(X) \) to be the ideals of the form \( O^A \), where \( A \) is a unique closed subset of \( \beta X \), and Abu Osba and Al-Ezeh [1, Theorem 3.2] proved that \( C_K(X) \) is pure if and only if \( \text{coz} C_K(X) = \bigcup_{f \in c_K} S(f) \).

In this paper, we characterize purity of \( C_{\Psi}(X) \) using the subspace \( kX \), the set of all points in \( \nu X \) having compact nbhds, then we use this characterization to study some algebraic properties of this ideal, such as projectivity, when the principal ideal \( (f) \) is projective or flat for each \( f \in C_{\Psi}(X) \). We found that if \( C_{\Psi}(X) \) is pure, then it is projective if and only if \( kX \) is paracompact, the principal ideal \( (f) \) is projective (flat) if and only if \( kX \) is basically disconnected (\( F' \)-space). An example is given to show that these results are false if \( C_{\Psi}(X) \) is not pure.

The following result is well known and is used very often in this article.

**Proposition 1.1.** For each space \( X \), \( C(X) \) is isomorphic to \( C(\nu X) \), and \( C_{\Psi}(X) \) is isomorphic to \( C_K(\nu X) \).

**Proof.** Let \( \varphi : C(X) \rightarrow C(\nu X) \) be defined such that \( \varphi(f) = f^\nu \). Then \( \varphi \) is the required isomorphism, see [5, Section 8.1] and [6, Theorem 2.1].

In this paper, we use the above proposition together with the results we obtained in [1] to characterize purity of the ideal \( C_{\Psi}(X) \) using the subspace \( kX \).

2. The subspace \( kX \). For each ideal \( I \) in \( C(X) \), define \( \theta(I) = \{ x \in \beta X : I \subseteq M^X \} \). Then \( \theta(I) = \bigcap_{f \in I} \text{cl}_\beta Z(f) \), see [5, Exercise 70.1].

Let \( kX = \beta X - \theta(C_{\Psi}(X)) = \{ x \in \beta X : C_{\Psi}(X) \) is not contained in \( M^X \} \). The space \( kX \) is important in classifying some properties of \( X \) and some of its extensions and it is related to the ideal \( C_K(\nu X) \). The following propositions and corollaries illustrate this fact.
PURITY OF THE IDEAL $C_Ψ(X)$

**Proposition 2.1** (see [6, Corollary 3.3 and Theorems 3.1 and 5.3]). The following statements are equivalent for any space $X$:

(i) $X$ is locally pseudocompact;
(ii) $X \subseteq kX$;
(iii) $ηX$ is locally compact;
(iv) $C_Ψ(X)$ is not contained in any fixed maximal ideal.

**Proposition 2.2** (see [6, Theorems 3.2, 5.1, and 5.2]). For each space $X$,

(i) $kX = \text{Int}_{βX} υX = \text{Int}_{βX} ΨX$;
(ii) $ΨX = X \cup kX$;
(iii) $ΨX − X = \bigcup_{f \in C_Ψ(X)} S(fυ) − S(f)$.

**Proposition 2.3** (see [1, Theorem 2.2]). For each space $X$,

$\text{coz} CK(X) = \text{Int} βXX$.

The following result is an easy consequence of Propositions 2.2 and 2.3.

**Corollary 2.4.** For each space $X$, $kX = \text{coz} CK(υX) = \bigcup_{f \in C_Ψ(X)} υX − Z(fυ)$.

**Corollary 2.5.** For each space $X$, $kX = \bigcup_{f \in C_Ψ(X)} βX − Z(fβ)$.

**Proof.** Let $f \in C_Ψ(X) \subseteq C^*(X)$. For each $p \in βX − υX$, $f \in M^p \cap C^*$.

So $fβ(p) = 0$ for each $p \in βX − υX$. Thus $Z(fβ) = (βX − υX) ∪ Z(fυ)$, and $βX − Z(fβ) = υX ∩ (βX − Z(fυ)) = υX − Z(fυ)$.

Now, $\bigcup_{f \in C_Ψ(X)} βX − Z(fβ) = \bigcup_{f \in C_Ψ(X)} υX − Z(fυ) = \bigcup_{fυ \in CK(υX)} υX − Z(fυ) = kX$, by Corollary 2.4.

**Theorem 2.6.** The space $ΨX$ is locally compact if and only if $X$ is locally pseudocompact and $θ(C_Ψ(X))$ is a round subset of $βX$.

**Proof.** See [6, Theorem 5.4] and [10, Theorem 3.3].

3. Purity of $C_Ψ(X)$. Here we characterize purity of $C_Ψ(X)$ using the subspace $kX$. But first we need some preliminaries.

**Proposition 3.1** (see [1, Theorem 3.2]). For each space $X$, the ideal $C_K(X)$ is pure if and only if $\text{coz} C_K(X) = \bigcup_{f \in C_K(X)} S(f)$.

It was proved in [11, Theorem 2.4] that an ideal $I$ in $C(X)$ is pure if and only if $I = O^A$ where $A$ is a unique closed subset of $βX$. In fact, it was proved that $A$ must be the set $\bigcap_{f \in \text{cl}_RX} Z(f) = θ(I)$. Here we show that if the ideal $O^A$ is pure, then $A$ need not be closed, but $O^A = O^{cl_{βX}A}$.

**Theorem 3.2.** The ideal $O^A$ is pure if and only if $O^A = O^{cl_{βX}A}$.

**Proof.** Suppose that $O^A$ is pure and $f \in O^A$. Then there exists $g \in O^A$ such that $f = fg$. So $fβ = fβgβ$ which implies that $S(fβ) \subseteq \text{coz} gβ$. Hence $A \subseteq \text{Int}_{RX} Z(gβ) \subseteq Z(gβ) \subseteq βX − S(fβ) \subseteq Z(fβ)$. Thus $\text{cl}_{RX} A \subseteq Z(gβ) \subseteq βX − S(fβ) \subseteq \text{Int}_{RX} Z(fβ)$ which implies that $f \in O^{cl_{βX}A}$.

In the following theorem we characterize purity of the ideal $C_Ψ(X)$ using properties of the subspace $kX$. 
Theorem 3.3. The following statements are equivalent:

1. $C_\Psi(X)$ is pure;
2. $C_\Psi(X) = O^{\beta X - kX}$;
3. $\beta X - kX$ is a round subset of $\beta X$.

Proof. (1)⇒(2). $C_\Psi(X)$ is pure if and only if $C_\Psi(X) = O^{\beta X - \omega X} = \mathcal{O}^{\beta X - kX} = O^{\beta X - \text{int}\,\beta X}$, see Proposition 2.2 and Theorem 3.2 above.

(2)⇒(3). $M^{\beta X - kX} \supseteq O^{\beta X - kX} = C_\Psi(X) = M^{\beta X - \omega X} \supseteq M^{\beta X - kX}$. So $\beta X - kX$ is a round subset of $\beta X$.

(3)⇒(1). $\mathcal{O}^{\beta X - \omega X} = \mathcal{O}^{\beta X - \text{int}\,\beta X} = \mathcal{O}^{\beta X - kX} = M^{\beta X - kX} = M^{\beta X - \omega X} = M^{\beta X - \omega X} = C_\Psi(X)$. So it follows by Theorem 3.2 that $C_\Psi(X)$ is pure.

The following result will be extremely useful throughout the rest of the paper.

Corollary 3.4. The ideal $C_\Psi(X)$ is pure if and only if for each $f \in C_\Psi(X)$, $S(f^0) \subseteq kX$.

Proof. The ideal $C_\Psi(X)$ is pure if and only if $C_k(\omega X)$ is pure if and only if for each $f \in C_\Psi(X)$, $S(f^0) \subseteq kX$, see Propositions 1.1 and 3.1.

Corollary 3.5. The space $\Psi X$ is locally compact if and only if $X \subseteq kX$ and $C_\Psi(X)$ is pure.

Proof. The result follows from Theorems 2.6 and 3.3.

It was shown in [1, Theorem 3.2] that $C_k(X)$ is pure if and only if $\text{coz} C_k(X) = \bigcup_{f \in C_k(X)} S(f)$. Now, if $C_\Psi(X)$ is pure, then it is easy to see that $$\text{coz} C_\Psi(X) = \bigcup_{f \in C_\Psi(X)} S(f).$$ (3.1)

This raises the following question: suppose that $\text{coz} C_\Psi(X) = \bigcup_{f \in C_\Psi(X)} S(f)$, does this imply that $C_\Psi(X)$ is pure? The following example shows that this need not be true.

Example 3.6. Let $W^* = [0, \omega_1]$ be the set of all ordinals less than or equal to the first uncountable ordinal number $\omega_1$. Let $W = [0, \omega_1)$ and $T^* = W^* \times \mathbb{N}^*$, where $\mathbb{N}^*$ denote the one point compactification $\mathbb{N} \cup \{\omega_0\}$ of the natural numbers. Let $t = (\omega_1, t_0)$, $T = T^* - \{t\}$, let $A = W \times \{\omega_0\}$ and let $B = \{\omega_1\} \times \mathbb{N}$. Let $S$ be obtained from $T \times \mathbb{N}$ by identifying $A \times \{2n - 1\}$ with $A \times \{2n\}$ and identifying $B \times \{2n\}$ with $B \times \{2n + 1\}$. Then $S$ is locally compact, since $T$ is, and $A \cap B = \emptyset$, see [7, Example 7.3] and [9, page 240]. Let $H$ be obtained from $T^* \times \mathbb{N}$ by identifying $(A \cup \{t\}) \times \{2n - 1\}$ with $(A \cup \{t\}) \times \{2n\}$ and identifying $(B \cup \{t\}) \times \{2n\}$ with $(B \cup \{t\}) \times \{2n + 1\}$. Now, $H$ is $\sigma$-compact and so it is realcompact. $H$ is not locally compact since $(\omega_1, \omega_0, n)$ has no compact neighborhood for each $n \in \mathbb{N}$. So $S \subseteq kS \subseteq \omega S \subseteq H$. Define $f : S \rightarrow \mathbb{R}$ by $f(\alpha, n, 1) = 1/n$ for all $\alpha \in W^*$, $n \in \mathbb{N}$ and by zero otherwise. Then $\text{coz} f = W^* \times \mathbb{N} \times \{1\}$ and $S(f) = T \times \{1\}$ is pseudocompact, noncompact. So $f \in C_\Psi(S) - C_k(S)$, which implies that $S$ is not $\Psi$-compact. Hence $kS = S \neq \Psi S$. Therefore, $\Psi S$ is not locally compact. So it follows by Corollary 3.5 that $C_\Psi(S)$ is not pure although $S$ is locally pseudocompact and $\text{coz} C_\Psi(S) = S = \bigcup_{f \in C_\Psi(S)} S(f)$.

4. Some applications. In this section, we use the characterization obtained in Theorem 3.3 and Corollary 3.4 above for purity of the ideal $C_\Psi(X)$ to characterize
when \( C_\Psi(X) \) is a projective \( C(X) \)-module, when every principal ideal of \( C_\Psi(X) \) is projective or flat \( C(X) \)-module, and for which spaces \( X \) and \( Y \), the two ideals \( C_\Psi(X) \) and \( C_\Psi(Y) \) are isomorphic.

**Theorem 4.1.** Let \( C_\Psi(X) \) and \( C_\Psi(Y) \) be pure ideals. Then \( C_\Psi(X) \) is isomorphic to \( C_\Psi(Y) \) if and only if \( kX \) is homeomorphic to \( kY \).

**Proof.** If \( C_\Psi(X) \) is isomorphic to \( C_\Psi(Y) \), then \( kX \) is homeomorphic to \( kY \), see [12, Corollary 4.11].

For the converse, we prove that \( C_K(\nu X) \) is isomorphic to \( C_K(\nu Y) \), then the result follows from **Proposition 1.1**.

Suppose \( \varphi : kX \to kY \) is a homeomorphism. Let \( f \in C_K(\nu Y) \), then \( f_1 \circ \varphi \in C(kX) \), where \( f_1 = f|_{kY} \). But \( \text{cos} f = \varphi(\text{cos}(f_1 \circ \varphi)) \), which implies that \( \varphi^{-1}(\text{cos} f) = \text{cos} (f_1 \circ \varphi) \).

Therefore \( \text{cl}_{kX} \text{cos}(f_1 \circ \varphi) = \text{cl}_{kX} \varphi^{-1}(\text{cos} f) = \varphi^{-1}(S(f)) \), since \( S(f) \) is contained in \( kY \) by the purity of \( C_\Psi(Y) \). Now, for each \( f \in C_K(\nu X) \) define

\[
\tilde{g}_f : \nu X \to \mathbb{R} \quad \text{by} \quad \tilde{g}_f(x) = \begin{cases} f_1 \circ \varphi(x), & x \in kX, \\ 0, & x \in \nu X - \varphi^{-1}(S(f)). \end{cases} \tag{4.1}
\]

Then, \( \tilde{g}_f \in C_K(\nu X) \), since \( S(\tilde{g}_f) = \text{cl}_{kX} \text{cos}(f_1 \circ \varphi) \) is compact.

Define \( \tilde{\varphi} : C_K(\nu Y) \to C_K(\nu X) \) by \( \tilde{\varphi}(f) = \tilde{g}_f \). Then \( \tilde{\varphi} \) is a ring homomorphism. It remains to show that \( \tilde{\varphi} \) is bijective.

To see that \( \tilde{\varphi} \) is one-to-one, suppose \( \tilde{\varphi}(f) = \tilde{\varphi}(g) = 0 \). Then \( f_1 \circ \varphi(x) = 0 \) for every \( x \in kX \). But \( \text{cos}(f_1 \circ \varphi) = \varphi^{-1}(\text{cos} f) \), and so \( \varphi^{-1}(\text{cos} f) = \emptyset \). Therefore \( f = 0 \).

To see that \( \tilde{\varphi} \) is onto, let \( f \in C_K(\nu X) \). Define

\[
g : \nu Y \to \mathbb{R} \quad \text{by} \quad g(y) = \begin{cases} f \circ \varphi^{-1}(y), & y \in kY, \\ 0, & y \in \nu Y - \varphi(S(f)). \end{cases} \tag{4.2}
\]

Then \( g \in C(\nu Y) \), since \( \varphi(S(f)) \) is compact. Here again we use the purity of \( C_\Psi(X) \), since we assumed that \( S(f) \subseteq kX \). Moreover, if \( g(y) \neq 0 \), then \( \varphi^{-1}(y) \in \text{cos} f \). So \( \text{cos} g \subseteq \varphi(\text{cos} f) \).

Thus, \( \text{cl}_{kY} \text{cos} g \subseteq \text{cl}_{kY} \varphi(\text{cos} f) = \varphi(\text{cl}_{\nu X} \text{cos} f) = \varphi(S(f)) \). Hence \( S(g) = \text{cl}_{kY} \text{cos} g \) is compact. It follows that \( g \in C_K(\nu Y) \).

Finally, note that

\[
\tilde{\varphi}(g)(x) = \begin{cases} g_1 \circ \varphi(x), & x \in kX, \\ 0, & x \in \nu X - \varphi^{-1}(S(g)). \end{cases} \tag{4.3}
\]

Then \( \tilde{\varphi}(g) = f \) and so \( \tilde{\varphi} \) is onto. Hence \( C_K(\nu X) \) is isomorphic to \( C_K(\nu Y) \). \( \square \)
Here we characterize when $C_{\overline{f}}(X)$ is a projective $C(X)$-module.

**Theorem 4.2.** The ideal $C_{\overline{f}}(X)$ is a projective $C(X)$-module if and only if $kX$ is paracompact and $C_{\overline{f}}(X)$ is pure.

**Proof.** It was proved by Brookshear [3, Theorem 3.10] that $C_k(X)$ is a projective $C(X)$-module if and only if $coz C_k(X)$ is paracompact and $S(f) \subseteq coz C_k(X)$ for each $f \in C_k(X)$. Our result now follows from Proposition 3.1 and Corollaries 2.4 and 3.4.

It was proved in [2, Lemma 2] and [3, Corollary 2.5] that the principal ideal $(f)$ is a projective (flat) $C(X)$-module if and only if $(f)$ is clopen in $X$. We can use this result to determine when the principal ideal $(f)$ is a projective or a flat $C(X)$-module for each $f \in C_{\overline{f}}(X)$.

**Theorem 4.3.** For each $f \in C_{\overline{f}}(X)$, the ideal $(f)$ is a projective $C(X)$-module if and only if $C_{\overline{f}}(X)$ is pure and $kX$ is basically disconnected.

**Proof.** Suppose that $kX$ is basically disconnected and $C_{\overline{f}}(X)$ is pure. Let $f \in C_{\overline{f}}(X)$. Then $S(f) \subseteq kX$ since $C_{\overline{f}}(X)$ is pure. Now let $f_i = f|_{kX}$ and note that $cl_{kX}\left(kX - Z(f_i)\right) = S(f_i)$. Since $kX$ is basically disconnected, $S(f)$ is open in $kX$ and therefore it is open in $\nu X$ (cf. Proposition 2.2). Thus $S(f) = S(f) \cap X$ is open in $X$. Hence the ideal $(f)$ is a projective $C(X)$-module.

Conversely, suppose that every principal ideal of $C_{\overline{f}}(X)$ is a projective $C(X)$-module. Then for each $f \in C_{\overline{f}}(X)$, $S(f)$ is open in $X$, so define

$$g(x) = \begin{cases} 1, & x \in S(f), \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

Then $g \in C_{\overline{f}}(X)$ and $f = fg$. Thus $C_{\overline{f}}(X)$ is a pure ideal.

To demonstrate basic disconnectedness, we first show that for each $f \in C_k(kX)$, $S(f)$ is clopen. Then we will use this result to show that for each $k \in C_k(kX)$, $S(k)$ is clopen.

Let $f \in C_k(kX)$. Then $f$ can be extended to a function $F \in C_k(\nu X)$ with $cl_{kX}(kX - Z(f)) = S(F)$ which is open, since $C_k(X)$ is isomorphic to $C_k(\nu X)$.

Now, let $k \in C(kX)$, and $a \in cl_{kX}(kX - Z(k)) \subseteq kX$. So there exists an open set $U$ such that $U$ is compact, and $a \in U \subseteq \bar{U} \subseteq kX$. There exists $f \in C(kX)$ such that $f(a) = 1$ and $f(kX - U) = 0$. Then $f \in C_k(kX)$.

Thus $a \in (kX - Z(f)) \cap cl_{kX}(kX - Z(k)) \subseteq cl_{kX}(kX - Z(f)) \cap cl_{kX}(kX - Z(k)) = cl_{kX}(kX - Z(f)) \cap (kX - Z(k)) = cl_{kX}(kX - Z(f)) \subseteq cl_{kX}(kX - Z(k))$. But $cl_{kX}(kX - Z(f))$ is compact, and so is clopen since $f$ is a pure ideal. So, $cl_{kX}(kX - Z(k))$ is clopen in $kX$. Thus $kX$ is basically disconnected.

**Theorem 4.4.** Let $X$ be a space such that $C_{\overline{f}}(X)$ is pure. Then for each $f \in C_{\overline{f}}(X)$ the principal ideal $(f)$ is a flat $C(X)$-module if and only if $kX$ is an $F'$-space.

**Proof.** Suppose that $kX$ is an $F'$-space, $f \in C_{\overline{f}}(X)$ and $g \in Ann(f)$. Let $f_1 = f|_{kX}$ and $g_1 = g|_{kX}$. Then $(kX - Z(f_1)) \cap (kX - Z(g_1)) = \emptyset$. So, $cl_{kX}(kX - Z(f_1)) \cap cl_{kX}(kX - Z(g_1)) = \emptyset$, since $kX$ is an $F'$-space. But $S(f) = cl_{kX}(kX - Z(f_1))$, since $C_{\overline{f}}(X)$ is
pure and $S(f^o) \subseteq kX$. Now if $x \in S(f^o)$, then $x \in cl_{kX}(kX - Z(f_1))$, which implies that $x \notin cl_{kX}(kX - Z(g_1))$. So there exists an open set $U \subseteq kX$ such that $x \in U$ and $U \cap (kX - Z(g_1)) = \emptyset$. Therefore $x \in U \subseteq Z(g_1) \subseteq Z(g^o)$. But $U$ is open in $\nu X$, since $kX$ is. So $U \cap (\nu X - Z(g^o)) = \emptyset$, which implies that $x \notin S(g^o)$. Thus $S(f^o) \cap S(g^o) = \emptyset$.

The compactness of $S(f^o)$ implies that there exists $k^o \in C(\nu X)$ such that $k^o(S(f^o)) = 0$ and $k^o(S(g^o)) = 1$. So, $k \in Ann(f)$, and $g = gk$. Thus the ideal $(f)$ is a flat $C(X)$-module since $Ann(f)$ is pure.

Conversely, suppose that the principal ideal $(f)$ is a flat $C(X)$-module for each $f \in C_{\Psi}(X)$. Let $g, k \in C(kX)$ such that $gk = 0$. Suppose $y \in cl_{kX}(kX - Z(g)) \cap cl_{kX}(kX - Z(k))$. There exists $f_1 \in C_k(\nu X)$ such that $f_1(y) \neq 0$. Let $f = f_1|_{kX}$, then $y \in cl_{kX}(kX - Z(fg)) \cap cl_{kX}(kX - Z(fk))$. Define

$$h_1(x) = \begin{cases} fg(x), & x \in cl_{kX}(kX - Z(fg)), \\ 0, & x \in \nu X - (kX - Z(fg)). \end{cases}$$

(4.5)

$$h_2(x) = \begin{cases} fk(x), & x \in cl_{kX}(kX - Z(fk)), \\ 0, & x \in \nu X - (kX - Z(fk)). \end{cases}$$

Then $h_1, h_2 \in C_k(\nu X)$, since $S(h_1)$ and $S(h_2)$ are compact sets. Moreover, $h_1h_2 = 0$. So, there exists $h'_1 \in Ann(h_2)$ such that $h_1 = h_1h'_1$. Hence $y \in cl_{kX}(kX - Z(fg)) = S(h_1) \subseteq coz h'_1$. But $h'_1(S(h_2)) = 0$, so $y \notin S(h_2) = cl_{kX}(kX - Z(fk))$, a contradiction. Hence $cl_{kX}(kX - Z(g)) \cap cl_{kX}(kX - Z(k)) = \emptyset$ and $kX$ is an $F^*$-space.

**Example 4.5.** Let $X = [-1, 1]$ with all its points isolated, except for $x = 0$ it has its usual nbhds. Then $X$ is regular, paracompact, and consequently realcompact. So $X = \nu X$ and $kX = X - \{0\} \subseteq X$.

Let

$$f(x) = \begin{cases} x, & x = \frac{1}{n}, n \in \mathbb{Z}^*, \\ 0, & \text{otherwise}. \end{cases}$$

(4.6)

Then $S(f) = \{1/n : n \in \mathbb{Z}^*\} \cup \{0\}$. So $f \in C_{\Psi}(X)$ and $S(f)$ is not contained in $kX$. So $C_{\Psi}(X)$ is not a pure ideal.

The set $S(f)$ is not open, so the ideal $(f)$ is not projective. $Ann(f)$ is not pure, since the function

$$g(x) = \begin{cases} 0, & x = \frac{1}{n}, n \in \mathbb{Z}^*, \\ x, & \text{otherwise}, \end{cases}$$

(4.7)

belongs to $Ann(f)$, but $S(g) = X - \{1/n : n \in \mathbb{Z}^*\}$ is not a subset of $coz Ann(f)$, since for each $h \in Ann(f)$, $h(0) = 0$. So the ideal $(f)$ is not a flat $C(X)$-module. Let $Y = X - \{0\}$, then $kX = Y = kY$, but $C_{\Psi}(X)$ is not isomorphic to $C_{\Psi}(Y)$, since the latter is pure.

This example shows that Theorems 4.1, 4.2, 4.3, and 4.4 need not be true if $C_{\Psi}(X)$ is not pure.
References


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