ON BRANCHWISE IMPLICATIVE BCI-ALGEBRAS

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We introduce a new class of BCI-algebras, namely the class of branchwise implicative BCI-algebras. This class contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras (Chaudhry, 1990), and the class of medial BCI-algebras. We investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative BCI-algebras.

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1. Introduction. Iséki and Tanaka [10] defined implicative BCK-algebras and studied their properties. Further, Iséki [7, 8] gave the notion of a BCI-algebra which is a generalization of the concept of a BCK-algebra. Iséki [8] and Iséki and Thaheem [11] have shown that no proper class of implicative BCI-algebras exists, that is, such BCI-algebras are implicative BCK-algebras.

Thus, a natural question arises whether it is possible to generalize the notion of implicativeness in such a way that this generalization not only gives us a proper class of BCI-algebras but also contains the class of implicative BCK-algebras. In this paper, we answer this question in yes by introducing the concept of a branchwise implicative BCI-algebra. This proper class of BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras [4, 6].

2. Preliminaries. A BCI-algebra is an algebra \((X, \cdot, 0)\) of type (2,0) satisfying the following conditions:

\[
(x \cdot y) \cdot (x \cdot z) \leq z \cdot y, \quad \text{where } x \leq y \text{ if and only if } x \cdot y = 0, \quad (2.1)
\]

\[
x \cdot (x \cdot y) \leq y, \quad (2.2)
\]

\[
x \leq x, \quad (2.3)
\]

\[
x \leq y \text{ and } y \leq x \text{ imply } x = y, \quad (2.4)
\]

\[
x \leq 0 \text{ implies } x = 0. \quad (2.5)
\]

If (2.5) is replaced by \(0 \leq x\), then the algebra is called a BCK-algebra. It is well known that every BCK-algebra is a BCI-algebra.

In a BCI-algebra \(X\), the following hold:

\[
(x \cdot y) \cdot z = (x \cdot z) \cdot y, \quad (2.6)
\]

\[
x \cdot 0 = x, \quad (2.7)
\]

\[
x \leq y \text{ implies } x \cdot z \leq y \cdot z \text{ and } z \cdot y \leq z \cdot x, \quad (2.8)
\]
\[(x \ast z) \ast (y \ast z) \leq x \ast y, \quad (2.9)\]
\[x \ast (x \ast (x \ast y)) = x \ast y \quad \text{(see [8]).} \quad (2.10)\]

**Definition 2.1** (see [9]). A subset \( I \) of a BCI-algebra \( X \) is called an ideal of \( X \) if it satisfies
\[0 \in I, \quad x \ast y \in I, \quad y \in I \text{ imply } x \in I. \quad (2.11)\]

**Definition 2.2** (see [10]). If in a BCK-algebra \( X \)
\[(x \ast y) \ast z = (x \ast z) \ast (y \ast z) \quad (2.12)\]
holds for all \( x, y, z \in X \), then it is called positive implicative.

**Definition 2.3** (see [10]). If in a BCK-algebra \( X \)
\[x \ast (x \ast y) = y \ast (y \ast x) \quad (2.13)\]
holds for all \( x, y \in X \), then it is called commutative.

**Theorem 2.4** (see [10]). A BCK-algebra \( X \) is positive implicative if and only if it satisfies
\[(x \ast y) = (x \ast y) \ast y \quad \forall x, y \in X. \quad (2.14)\]

It has been shown in [8, 11] that no proper classes of positive implicative BCI-algebras and commutative BCI-algebras exist and such BCI-algebras are BCK-algebras of the corresponding type. That is why we generalized these notions and defined weakly positive implicative BCI-algebras [1] and branchwise commutative BCI-algebras [3] and studied some of their properties. Each class of these proper BCI-algebras contains the class of BCK-algebras of the corresponding type.

**Definition 2.5** (see [1]). A BCI-algebra \( X \) satisfying
\[(x \ast y) \ast z = ((x \ast z) \ast z) \ast (y \ast z) \quad \forall x, y, z \in X \quad (2.15)\]
is called a weakly positive implicative BCI-algebra.

**Theorem 2.6** (see [1]). A BCI-algebra \( X \) is weakly positive implicative if and only if
\[x \ast y = ((x \ast y) \ast y) \ast (0 \ast y) \quad \forall x, y \in X. \quad (2.16)\]

A BCI-algebra satisfying \((x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u)\) is called a medial BCI-algebra.

Let \( X \) be a BCI-algebra and \( M = \{x : x \in X \text{ and } 0 \ast x = 0\} \). Then \( M \) is called its BCK-part. If \( M = \{0\} \), then \( X \) is called \( p \)-semisimple.

It has been shown in [4, 5, 6, 13] that in a BCI-algebra \( X \) the following are equivalent:
\[X \text{ is medial, } x \ast (x \ast y) = y \quad \forall x, y \in X, \quad (2.17)\]
\[0 \ast (0 \ast x) = x \quad \forall x \in X, \quad X \text{ is } p \text{-semisimple.}\]

We now describe the notions of branches of a BCI-algebra and branchwise commutative BCI-algebras defined and investigated in [2, 3].
**Definition 2.7** (see [3]). Let $X$ be a BCI-algebra, then the set $\text{Med}(X) = \{x: x \in X$ and $0 \ast (0 \ast x) = x\}$ is called medial part of $X$.

Obviously, $0 \in \text{Med}(X)$ and thus $\text{Med}(X)$ is nonempty. In what follows the elements of $\text{Med}(X)$ will be denoted by $x_0, y_0, \ldots$. It is known that $\text{Med}(X)$ is a medial sub-algebra of $X$ and for each $x \in X$, there is a unique $x_0 = 0 \ast (0 \ast x) \in \text{Med}(X)$ such that $x_0 \leq x$ (see [3]). Further, $\text{Med}(X)$, in general, is not an ideal of $X$. Obviously, for a BCK-algebra $X$, $\text{Med}(X) = \{0\}$ and hence is an ideal of $X$.

**Definition 2.8** (see [3]). Let $X$ be a BCI-algebra and $x_0 \in \text{Med}(X)$, then the set $B(x_0) = \{x: x \in X \text{ and } x_0 \ast x = 0\}$ is called a branch of $X$ determined by the element $x_0$.

The following theorem (proved in [2, 3]) shows that the branches of a BCI-algebra $X$ are pairwise disjoint and form its partition. So the study of branches of a BCI-algebra $X$ plays an important role in investigation of the properties of $X$. Obviously, a BCK-algebra $X$ is a one-branch BCI-algebra and in this case $X = B(0)$.

**Theorem 2.9** (see [2, 3]). Let $X$ be a BCI-algebra with medial part $\text{Med}(X)$, then

(i) $X = \cup \{B(x_0): x_0 \in \text{Med}(X)\}$.

(ii) $B(x_0) \cap B(y_0) = \emptyset$, $x_0, y_0 \in \text{Med}(X)$, and $x_0 \neq y_0$.

(iii) If $x, y \in B(x_0)$, then $0 \ast x = 0 \ast y = 0 \ast x_0 = 0 \ast y_0$ and $x \ast y, y \ast x \in M$.

**Definition 2.10** (see [3]). A BCI-algebra $X$ is said to be branchwise commutative if and only if for $x_0 \in \text{Med}(X)$, $x, y \in B(x_0)$, the following equality holds:

$$x \ast (x \ast y) = y \ast (y \ast x).$$ (2.18)

Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is commutative if and only if it is branchwise commutative.

**Theorem 2.11** (see [3]). A BCI-algebra $X$ is branchwise commutative if and only if

$$x \ast (x \ast y) = y \ast (y \ast (x \ast y)) \quad \forall x, y \in X.$$ (2.19)

3. **Branchwise implicative BCI-algebras.** In this section, we define branchwise implicative BCI-algebras. We show that this proper class of BCI-algebras contains the class of implicative BCK-algebras [10], the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras. We also find necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.

**Definition 3.1** (see [10]). A BCK-algebra $X$ is said to be implicative if and only if

$$x \ast (y \ast x) = x \quad \forall x, y \in X.$$ (3.1)

It has been shown in [8, 11] that no proper class of implicative BCI-algebras exists. Due to this reason we generalized the notion of implicativeness to weak implicativeness [1] mentioned below.

**Definition 3.2** (see [1]). A BCI-algebra $X$ is said to be weakly implicative if and only if

$$x = (x \ast (y \ast x)) \ast (0 \ast (y \ast x)) \quad \forall x, y \in X.$$ (3.2)
We further generalize this concept and find a generalization of the following well-known result of Iséki [10].

**Theorem 3.3.** An implicative BCK-algebra is a positive implicative and commutative BCK-algebra.

**Definition 3.4.** A BCI-algebra $X$ is said to be a branchwise implicative BCI-algebra if and only if

$$x \ast (y \ast x) = x \quad \forall x, y \in B(x_0) \text{ and } x_0 \in \text{Med}(X).$$

(3.3)

**Example 3.5.** Let $X = \{0, 1, 2\}$ in which $\ast$ is defined by

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Then $X$ is a branchwise implicative BCI-algebra. This shows that proper branchwise implicative BCI-algebras exist.

**Remark 3.6.** (i) Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is implicative if and only if it is branchwise implicative.

(ii) Let $X$ be weakly implicative and let $x, y \in B(x_0), x_0 \in \text{Med}(X)$, then using Theorem 2.9(iii), we get $y \ast x \in M$. Thus $0 \ast (y \ast x) = 0$. So $x = (x \ast (y \ast x)) \ast (0 \ast (y \ast x))$ reduces to $x = x \ast (y \ast x)$. Hence every weakly implicative BCI-algebra is branchwise implicative BCI-algebra. But the branchwise implicative BCI-algebra $X$ of Example 3.5 is not weakly implicative because $(1 \ast (2 \ast 1)) \ast (0 \ast (2 \ast 1)) = (1 \ast 2) \ast (0 \ast 2) = 2 \ast 2 = 0 \neq 1$.

(iii) It is known that each branch of a medial BCI-algebra $X$ is a singleton. Let $X$ be a medial BCI-algebra and $x_0 \in \text{Med}(X)$. Then $B(x_0) = \{x_0\}$. Hence $x_0 \ast (x_0 \ast x_0) = x_0 \ast 0 = x_0$, which implies that $X$ is branchwise implicative.

Thus the class of branchwise implicative BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras, and the class of medial BCI-algebras. We now prove the following results.

**Lemma 3.7.** Let $X$ be a BCI-algebra. If $x, y \in X$ and $x \leq y$, then $x, y \in B(x_0)$ for $x_0 \in \text{Med}(X)$.

**Proof.** Let $x \in X$, then there is a unique $x_0 = 0 \ast (0 \ast x) \in \text{Med}(X)$ such that $x \in B(x_0)$. Now $x_0 \ast y = (0 \ast (0 \ast x)) \ast y = (0 \ast y) \ast (0 \ast x) \leq x \ast y = 0$. Hence $x_0 \ast y = 0$, which implies $y \in B(x_0)$. \hfill $\square$

**Theorem 3.8.** If $X$ is a branchwise implicative BCI-algebra, then it is branchwise commutative.

**Proof.** Let $x, y \in X$, then $x \ast (x \ast y) \leq y$ and Lemma 3.7 imply that $x \ast (x \ast y)$ and $y \in B(y_0)$ for some $y_0 \in \text{Med}(X)$. Since $X$ is branchwise implicative, therefore
using (3.3), we get
\[(x * (x * y)) * (y * (x * (x * y))) = x * (x * y).\] (3.4)

Using (2.2) and (2.8), we get
\[x * (x * y) = (x * (x * y)) * (y * (x * (x * y))) \leq y * (y * (x * (x * y))) \leq x * (x * y).\] (3.5)

Thus
\[x * (x * y) = y * (y * (x * (x * y))),\] (3.6)

which along with Theorem 2.11 implies that \(X\) is branchwise commutative. 

**Theorem 3.9.** If \(X\) is a branchwise implicative BCI-algebra, then it satisfies
\[(x * y) * (0 * y) = ((x * y) * y) * (0 * y).\] (3.7)

**Proof.** Since \(X\) is branchwise implicative, therefore Theorem 3.8 implies that \(X\) is branchwise commutative. Let \(x, y \in X\). Since \((x * y) * (0 * y) \leq x\), therefore Lemma 3.7 implies that \((x * y) * (0 * y), x \in B(x_0)\). Now branchwise implicativeness of \(X\) implies
\[
((x * y) * (0 * y)) * (x * ((x * y) * (0 * y))) = (x * y) * (0 * y),
\] (3.8)

which, using (2.6) twice, gives
\[
((x * ((x * (x * y) * (0 * y))) * y)) * (0 * y) = (x * y) * (0 * y).
\] (3.9)

Using branchwise commutativeness of \(X\), from (3.9) we get
\[
(((x * y) * (0 * y)) * ((x * y) * (0 * y)) * x)) * y) * (0 * y) = (x * y) * (0 * y), \] (3.10)

which implies
\[
(((x * y) * (0 * y)) * y) * (0 * y) = (x * y) * (0 * y),
\] (3.11)

so
\[
(((x * y) * y) * (0 * y)) * (0 * y) = (x * y) * (0 * y). \] (3.12)

**Remark 3.10.** Since a BCK-algebra is a one-branch BCI-algebra, therefore an implicative BCK-algebra is commutative. Further, for a BCK-algebra \(0 * y = 0\) and thus (3.7) reduces to \(x * y = (x * y) * y\), which implies \(X\) is positive implicative. So we get Theorem 3.3, a well-known result of Iséki [10], as a corollary from Theorems 3.8 and 3.9.

We now investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.
Theorem 3.11. A BCI-algebra \( X \), with \( \text{Med}(X) \) as an ideal of \( X \), is a branchwise implicative BCI-algebra if and only if it is branchwise commutative and satisfies

\[
(x \ast y) \ast (0 \ast y) = (((x \ast y) \ast y) (0 \ast y)) \ast (0 \ast y) \quad \forall x, y \in X. \tag{3.13}
\]

Proof. \((\Rightarrow)\) Sufficiency follows from Theorems 3.8 and 3.9.

\((\Leftarrow)\) For necessity we consider \( x, y \in X \) such that \( x, y \in B(x_0) \) for some \( x_0 \in \text{Med}(X) \). Now from Theorem 2.9(iii), we get \( x \ast y \) and \( y \ast x \in M \). So \( 0 \ast (x \ast y) = 0 \ast (y \ast x) = 0 \). Further, \( (x \ast (y \ast x)) \ast x = (x \ast x) \ast (y \ast x) = 0 \ast (y \ast x) = 0 \), so \( x \ast (y \ast x) \leq x \).

Now (3.14) along with Lemma 3.7 implies \( x \ast (y \ast x) \) and \( x \) belong to the branch determined by \( x \), that is, \( B(x_0) \). Hence \( x, y \) and \( x \ast (y \ast x) \in B(x_0) \). Since \( X \) is branchwise commutative, therefore,

\[
(x \ast (x \ast (y \ast x))) \ast (0 \ast x)
= [((y \ast x) \ast (x \ast (y \ast x))) \ast (0 \ast x)] \ast x
= [[((y \ast x) \ast (x \ast (y \ast x))) \ast (x \ast (y \ast x))) \ast (x \ast (y \ast x))] \ast (0 \ast x) \ast (0 \ast x). \tag{3.15}
\]

Now by using (2.6) three times, we get

\[
(x \ast (x \ast (y \ast x))) \ast (0 \ast x)
= [[[((y \ast x) \ast (0 \ast x)) \ast (0 \ast x)] \ast (0 \ast x)] \ast (0 \ast x). \tag{3.16}
\]

Since \( x, y \) and \( x \ast (y \ast x) \in B(x_0) \), therefore \( x \ast y, y \ast x, x \ast (x \ast (y \ast x)) \in M = B(0) \). Since \( X \) is branchwise commutative, therefore,

\[
(x \ast (x \ast (y \ast x))) \ast (0 \ast x)
= [(((x \ast (x \ast (y \ast x))) \ast (x \ast (y \ast x))) \ast (y \ast x)) \ast (0 \ast x)] \ast (0 \ast x)
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= ((0 \ast (y \ast x))) \ast (0 \ast x)
= (0 \ast (0 \ast x)) \ast (0 \ast x)
= (0 \ast x) \ast (0 \ast x) = 0 \ast (0 \ast x). \tag{3.17}
\]

Hence

\[
(x \ast (x \ast (y \ast x))) \ast (0 \ast x) = 0 \ast (0 \ast x) \in \text{Med}(X). \tag{3.18}
\]
But (2.10) implies 0 \ast (0 \ast (0 \ast x)) = 0 \ast x. So 0 \ast x \in \text{Med}(X). Since \text{Med}(X) is an ideal of X, therefore, x \ast (x \ast (y \ast x)) \in \text{Med}(X). Hence
\[ x \ast (x \ast (y \ast x)) = 0 \ast (0 \ast (x \ast (x \ast (y \ast x)))) . \tag{3.19} \]

Since x \ast (x \ast (y \ast x)) \in M = B(0), therefore, 0 \ast (x \ast (x \ast (y \ast x))) = 0. Thus x \ast (x \ast (y \ast x)) = 0, which gives
\[ x \leq x \ast (y \ast x) . \tag{3.20} \]

Using (3.14) and (3.20), we get
\[ x = x \ast (y \ast x) \quad \forall x, y \in B(x_0) . \tag{3.21} \]

Hence X is branchwise implicative. This completes the proof.

**Remark 3.12.** Since in a BCK-algebra X, Med(X) = \{0\} is always an ideal of X, therefore the following well-known result regarding BCK-algebra follows as a corollary from Theorem 3.11.

**Corollary 3.13.** A BCK-algebra is implicative if and only if it is positive implicative and commutative.

**Remark 3.14.** The following example shows that there exist proper BCI-algebras in which Med(X) is an ideal. Thus the condition, Med(X) is an ideal of X, in Theorem 3.11 is not unnatural.

**Example 3.15** (see [12, Example 2]). The set X = \{0, 1, 2, 3\} with the operation \ast defined as

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 2 & 2 \\
1 & 1 & 0 & 3 & 2 \\
2 & 3 & 3 & 2 & 0 \\
3 & 1 & 1 & 0 & 0 \\
\end{array}
\]

is a proper BCI-algebra. Here Med(X) = \{0, 2\} is an ideal of X. Further, X is branchwise implicative but is not medial.

**Definition 3.16.** Let X be a BCI-algebra. Two elements x, y of X are said to be comparable if and only if either x \ast y = 0 or y \ast x = 0, that is, either x \leq y or y \leq x.

**Definition 3.17.** Let X be a BCI-algebra. If x_0 \in \text{Med}(X) and x_0 \neq 0, then B(x_0), the branch of X determined by x_0, is called a proper BCI-branch of X.

**Theorem 3.18.** Let X be a BCI-algebra such that any two elements of a proper BCI-branch of X are comparable. Then X is branchwise implicative if and only if X is branchwise commutative and satisfies
\[ (x \ast y) \ast (0 \ast y) = (((x \ast y) \ast y) \ast (0 \ast y)) \ast (0 \ast y) \quad \forall x, y \in X . \tag{3.22} \]
Proof. \((\Rightarrow)\) Sufficiency follows from Theorems 3.8 and 3.9. 
\((\Leftrightarrow)\) For necessity we consider the following two cases.

**Case 1.** Let \(x, y \in B(0) = M\). Then \(0 \ast y = 0 \ast x = 0\) and hence (3.22) becomes \(x \ast y = (x \ast y) \ast y\). Further, \((x \ast (y \ast x)) \ast x = (x \ast x) \ast (y \ast x) = 0 \ast (y \ast x) = 0\). Hence

\[
x \ast (y \ast x) \leq x.
\]

(3.23)

Since \(x \ast y \in M = B(0)\) and \(X\) is branchwise commutative, therefore,

\[
x \ast (x \ast (y \ast x)) = (y \ast x) \ast ((y \ast x) \ast x) = (y \ast x) \ast (y \ast x) = 0.
\]

(3.24)

Thus

\[
x \ast \leq x \ast (y \ast x).
\]

(3.25)

From (3.23) and (3.25), we get \(x = x \ast (y \ast x)\) for all \(x, y \in B(0)\).

**Case 2.** Let \(x, y \in B(x_0)\), where \(x_0 \in Med(X)\) and \(x_0 \neq 0\). Thus \(x \ast y \in M\) and \(y \ast x \in M\). So \(0 \ast (x \ast y) = 0\) and \(0 \ast (y \ast x) = 0\). Further, taking \(y = x \ast y\) in (3.22), we get

\[
x \ast (x \ast y) = (x \ast (x \ast y)) \ast (x \ast y) \quad \forall x, y \in B(x_0).
\]

(3.26)

Interchanging \(x\) and \(y\) in (3.26), we get

\[
y \ast (y \ast x) = (y \ast (y \ast x)) \ast (y \ast x) \quad \forall x, y \in B(x_0).
\]

(3.27)

Since \(x, y\) are comparable, therefore, either \(y \ast x = 0\) or \(x \ast y = 0\). If \(y \ast x = 0\), then

\[
x \ast (y \ast x) = x \ast 0 = x.
\]

(3.28)

If \(x \ast y = 0\), then branchwise commutativity of \(X\) gives

\[
y \ast (y \ast x) = x \ast (x \ast y) = x \ast 0 = x.
\]

(3.29)

Using (3.27) and (3.29), we get

\[
x = x \ast (y \ast x).
\]

(3.30)

Thus \(X\) is branchwise implicative.

**Remark 3.19.** The following example shows that the conditions \(Med(X)\) is an ideal of \(X\) and any two elements of a proper BCI-branch of \(X\) are comparable cannot be removed from Theorems 3.11 and 3.18, respectively.

**Example 3.20.** Let \(X = \{0, 1, 2, 3, 4, 5\}\) in which \(*\) is defined by

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Thus

\[
\begin{align*}
0 & \rightarrow 3 \\
3 & \rightarrow 1 \\
1 & \rightarrow 2 \\
2 & \rightarrow 5 \\
5 & \rightarrow 4 \\
4 & \rightarrow 0
\end{align*}
\]
Routine calculations give that $X$ is a BCI-algebra, which is branchwise commutative and satisfies (3.22). But we note that

1. Med$(X) = \{0, 2, 3\}$ is not an ideal of $X$ because $4 \ast 3 = 3 \in$ Med$(X)$, $3 \in$ Med$(X)$ but $4 \notin$ Med$(X)$. Further, $X$ is not branchwise implicative because $4, 5 \in B(2)$ and $4 \ast (5 \ast 4) = 4 \ast 1 = 2 \neq 4$;

2. the elements 4 and 5 of $B(2)$ are not comparable and also $X$ is not branchwise implicative.

Combining Theorems 3.11 and 3.18, we get the following theorem.

**Theorem 3.21.** Let $X$ be a BCI-algebra such that either Med$(X)$ is an ideal of $X$ or every pair of elements of a proper BCI-branch of $X$ are comparable, then $X$ is branchwise implicative if and only if $X$ is branchwise commutative and satisfies (3.22).

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**References**


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