MULTIPLIERS ON $L(S)$, $L(S)^{**}$, AND $LUC(S)^*$ FOR A LOCALLY COMPACT TOPOLOGICAL SEMIGROUP

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We study compact and weakly compact multipliers on $L(S)$, $L(S)^{**}$, and $LUC(S)^*$, where the latter is the dual of $LUC(S)$. We show that a left cancellative semigroup $S$ is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on $L(S)^{**}$. We also prove that $S$ is left amenable if and only if there is a nonzero compact (or weakly compact) multiplier on $LUC(S)^*$.

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1. Introduction. Let $S$ be a locally compact, Hausdorff topological semigroup. Let $M(S)$ be the space of all complex Borel measures on $S$. It is known that $M(S) = C_0(S)^*$, therefore, $M(S)$ is a Banach space and with convolution $\mu \ast \nu(\psi) = \int \int \psi(xy) d\mu(x) d\nu(y)$ ($\mu, \nu \in M(S), \psi \in C_0(\psi))$, $M(S)$ is a Banach algebra. The subalgebra $L(S)$ of $M(S)$ is defined by $L(S) = \{\mu \in M(S) \mid x \rightarrow |\mu| \ast \delta_x \ast |\mu| \text{ from } S \text{ to } M(S) \text{ are norm continuous}\}$ [1]. A semigroup $S$ is called foundation if $S = \bigcup_{\mu \in L(S)} \text{supp} \mu$. A trivial example is a topological group and in this case $L(S) = L^1(G)$. Let $C_b(S)$ be the set of all bounded continuous function on $S$. Let $LUC(S)^* = \{f \in C_b(S) \mid x \rightarrow l_x f \text{ is norm continuous}\}$ and $RUC(S)^* = \{f \in C_b(S) \mid x \rightarrow r_x f \text{ is norm continuous}\}$ where $l_x f(y) = f(xy)$, $r_x f(y) = f(yx)$. When $S$ is foundation, it is known that $L(S)$ has a bounded approximate identity [1], and therefore, the multiplier algebra of $L(S)$ is $M(S)$ [4]. Let $L(S)^*$ and $L(S)^{**}$ be the first and second duals of $L(S)$ and similarly, $M(S)^*$ and $M(S)^{**}$ be the first and second duals of $M(S)$. We also use the notation $LUC(S)^*$, $RUC(S)^*$ for the duals of $LUC(S)$, and $RUC(S)$, respectively. The subalgebras $LUC(S)$ and $RUC(S)$ are Banach C*-subalgebras of $C_b(S)$. With Arens product, $L(S)^{**}$ and $M(S)^{**}$ are Banach algebra. Also, with the same type product $LUC(S)^*$ is a Banach algebra. In this paper, among other things, we show that when $S$ is a left cancellative foundation semigroup, then $S$ is left (right) amenable if and only if there is a nonzero left (right) compact or weakly compact multiplier on $L(S)^{**}$ (or $LUC(S)^*$).

2. Preliminaries. For a Banach algebra $A$, we denote by $A^*$ and $A^{**}$ the first and second dual of $A$, respectively. On $A^{**}$ we define the first Arens product by

$$\langle mn, f \rangle = \langle m, nf \rangle, \quad \langle nf, a \rangle = \langle n, fa \rangle, \quad \langle fa, b \rangle = f(ab)$$ (2.1)

($m, n \in A^{**}; \ f \in A^*; \ a, b \in A$). With this product $A^{**}$ is a Banach algebra. We can also define a similar product on $LUC(S)^*$ such that $\langle mn, f \rangle = \langle m, nf \rangle$, $nf(x) = n(l_x f)$, $l_x f(y) = f(xy)$ ($m, n \in LUC(S)^*; \ f \in LUC(S); \ x, y \in S$). Clearly, $LUC(S)^*$ is a Banach algebra. A linear map on a Banach algebra $A$ is called a multiplier if
Therefore, \( π: \delta x(ψ) \) for topological groups. Then the following are equivalent:

- \( \langle \pi \rangle_{\text{topology and weak compact}} \]
- \( \langle \pi \rangle_{\text{onto (by the Hahn-Banach theorem)}} \]
- \( \langle \pi \rangle_{\text{onto (by the Hahn-Banach theorem)}} \)

Following Ghahramani and Lau [3], we have the following lemma (see [3, Lemmas 1.1, 1.2, 1.4, Proposition 1.3]).

**Lemma 2.1.** (a) Let \( f \in A^* \), \( b \in B \). Then \( b \pi(f) = π(i(b)f) \).

(b) The mapping \( π^*: B^{**} \to A^{**} \) is a homeomorphism whenever \( B^{**} \) has the weak*-topology and \( π^*(B^{**}) \) has the relative weak*-topology.

**Lemma 2.2.** Let \( B \) be a closed ideal in \( A \), \( n \in A^{**} \). If \( (a_α) \) is a bounded net in \( A \) such that \( a_α \to n \), then \( i(b)a_α \to\) \( π^*(b)n \) \( (b \in B) \).

**Proposition 2.3.** Let \( B \) be a right (or left) ideal of \( A \). Then \( π^*(B^{**}) \) is a right (resp., left) ideal of \( A^{**} \).

**Lemma 2.4.** Let \( A \) be a commutative Banach algebra. Then any weak*-closed right ideal in \( A^{**} \) is an ideal. If \( X = \text{spec } A \), then \( h(n) = \langle n, δ_x \rangle \) is a multiplicative on \( A^{**} \), where \( δ_x(ψ) = \langle x, ψ \rangle \).

3. **Multipliers on \( LUC(S)^* \) and \( L(S)^{**} \).** First we prove a theorem which is new even for topological groups.

**Theorem 3.1.** Let \( S \) be a right cancellative topological semigroup with identity \( e \). Then the following are equivalent:

- (a) \( S \) is left amenable.
- (b) There is a nonzero compact (or weakly compact) right multiplier on \( LUC(S)^* \).

**Proof.** (a)\(\Rightarrow\)(b). Let \( S \) be left amenable and \( m \) be a left invariant mean on \( LUC(S) \). Then \( \langle nm, f \rangle = \langle n, mf \rangle \), \( mf(x) = m(l_x(f)) = m(f) \) \( (f \in LUC(S)^*, \ f \in LUC(S)) \). Therefore, \( \langle nm, f \rangle = \langle n, m(f) \rangle = m(f)\langle n, 1 \rangle \), that is, \( nm = \langle n, 1 \rangle m \). Thus \( l_m(n) = \langle n, 1 \rangle m \) is a rank one operator and hence compact.

(b)\(\Rightarrow\)(a). Let \( T \) be a nonzero weakly compact right multiplier on \( LUC(S)^* \). Then \( T(m) = T(mδ_e) = mT(δ_e) = l_T(δ_e)m \). So, \( T = l_n \) where \( n = T(δ_e) \). Note that \( δ_e \in \text{LUC}(S)^* \) and \( δ_e(f) = f(e) \) \( (f \in \text{LUC}(S)^*) \). Now, let \( A = \{δ_xn \mid x \in S \} = \{δ_xT(δ_e) \mid x \in S \} = \{T(δ_x) \mid x \in S \} \) which is weakly compact. By Krein-Smulian's theorem \( K = \text{co} A \) is weakly compact [2]. Now, we show that if \( k \neq k' \in K \), then \( \|δ_xk \| \leq \|k_1\| \). On the other hand, if we define

\[
g(y) = \begin{cases} f(t), & y = tx, \\ 0, & \text{otherwise,} \end{cases}
\]

(3.1)
then \( g \) is well defined and belongs to \( \beta(S) \) (the space of bounded functions on \( S \)), then 
\( \delta_x g(t) = \delta_x (l_x g) = g(tx) = r_x g(t) = f(t) \). Let \( \hat{k}_1 \) be the extension of \( k_1 \) to \( \beta(S) \) (by the Hahn-Banach theorem). Then

\[
\|k_1\| = \|\hat{k}\| \leq \sup \{ \|\langle \hat{k}_1, f \rangle \| : f \in \beta(S) \}
\]

\[
= \sup \{ \|\langle \hat{k}_1, \delta_x g \rangle \| : g \in \beta(S) \}
\]

\[
= \sup \{ \|\langle \delta_x \hat{k}_1, g \rangle \| : g \in \beta(S) \}
\]

\[
= \|\delta_x k_1\| \quad \quad (3.2)
\]

It follows that \( \|\delta_x k_1\| = \|k_1\| \neq 0 \). Now, we show that if \( k, k' \in \text{co}(A) \), and \( k \neq k' \), then a similar argument shows that \( \|\delta_x (k - k')\| \neq 0 \). Finally, we show that \( 0 \notin \{ \delta_x (k - k') \mid x \in S \} \), hence, by a completely similar argument, we have \( \|\delta_x \alpha (k - k')\| = \|k - k'\| \neq 0 \).

Therefore, \( 0 \notin \{ \delta_x (k - k') \mid x \in S \} \). Hence, by Ryll-Nardzewski fixed point theorem [2], there exists a point \( q \in K \) such that \( \delta_x q = q \). It follows that \( \delta_x |q| = |\delta_x q| = |q| \), and \( \|q\| = \|n\| \neq 0 \). Now, if we take \( m = |q|/\|q\| \), then clearly \( \delta_x m = m \), so, \( m(f) = \delta_x m(f) = \delta_x (m f) = m f(x) = m(\chi f) \). Therefore, \( m \) is a left invariant mean on \( LUC(S) \), that is, \( S \) is left amenable.

For a foundation semigroup \( S \), let \( i : LUC(S) \to L(S)^* \) be such that \( \langle i(f), \mu \rangle = \langle \mu, f \rangle \) (\( f \in LUC(S) \), \( \mu \in L(S) \)) is an embedding and \( \pi = i^* : L(S)^* \to LUC(S)^* \) is onto. It is clear from the proof of [3, Lemma 2.2] for topological groups that \( \pi(E) = \delta_e \) where \( E \) is a right identity, \( \pi \) is a homomorphism and \( FG = F \pi(G) \). Also we have the following proposition which is similar to [6, Theorem 2.3].

We prove the following proposition for foundation semigroups with identity \( e \).

**Proposition 3.2.** Let \( E \) be a right identity in \( L(S)^* \). Then \( \pi \) is an isometric isomorphism of \( EL(S)^* \) onto \( LUC(S)^* \).

**Proof.** Let \( I \) be the identity operator on \( L(S)^* \). Then

\[
L(S)^* = EL(S)^* + (I - E)L(S)^* \quad \quad (3.3)
\]

Now, if \( m \in L(S)^* \), then \( \pi((I - E)m) = \pi(m) - \pi(E)\pi(m) = \pi(m) - \delta_e \pi(m) = \pi(m) - \pi(m) = 0 \). Thus \( (I - E)m \in \ker \pi \). On the other hand, if \( m \in \ker \pi \), then \( Em = E\pi(m) = 0 \). So \( m = m - Em = (I - E)m \in (I - E)L(S)^* \). Thus,

\[
\ker \pi = (I - E)L(S)^* \quad \quad (3.4)
\]

So, we have

\[
L(S)^* = EL(S)^* + \ker \pi \quad \quad (3.5)
\]

It follows that \( \pi \) is injective from \( EL(S)^* \) onto \( L(S)^*/\ker \pi \), therefore \( \pi \) is injective from \( EL(S)^* \) onto \( LUC(S)^* \), and so \( \pi \) is an algebra isomorphism. We also have \( \|Em\| = \|E\pi(m)\| \leq \|E\|\|\pi(m)\| = \|\pi(m)\| \leq \|m\| \), since \( \pi \) is a quotient map. Thus \( \|\pi(Em)\| \leq \|\pi\|\|Em\| \leq \|Em\| \leq \|\pi(m)\| \). So \( \|\pi(Em)\| = \|\pi(m)\| = \|Em\| \), that is, \( \pi \) is an isometry. \( \square \)
Now, we have another main theorem.

**Theorem 3.3.** Let $S$ be a right cancellative locally compact foundation semigroup with identity $e$. Then the following are equivalent:

(a) $S$ is left amenable.
(b) There is a nonzero compact (or weakly compact) right multiplier on $L(S)^{**}$.

**Proof.** (a)$\Rightarrow$(b). The proof of this part exactly reads the same line of the proof of (a)$\Rightarrow$(b) of Theorem 3.1, so it is omitted.

(b)$\Rightarrow$(a). Let $T$ be a nonzero weakly compact right multiplier on $L(S)^{**}$, so $T = l_n$ for some $n \in L(S)^{**}$. Now $l_{En}$ is also a nonzero right multiplier on $E L(S)^{**}$ where $E$ is a right identity of $L(S)^{**}$ with norm 1, since $l_{En}(Em) = EmEn = Emn$. Now by Proposition 3.2, $\pi(EL(S)^{**}) = (LUC(S))^*$ is isometrically isomorphic. If we define $l'_n = l_{En} \circ \pi$, then $l'_n$ is a nonzero right multiplier on $LUC(S)^*$. Therefore, $S$ is left amenable. \hfill $\Box$

In [3, Theorem 2.1] it was also shown that a locally compact group $G$ is amenable if and only if there is a nonzero compact (weakly compact) right multiplier on $M(G)^{**}$. But we are not able to extend this result to $M(S)^{**}$.

**Proposition 3.4.** A right multiplier $l_n(m) = mn$ $(m \in LUC(S)^*)$ is compact if and only if the restriction of $l_n$ to $M(S)$ is compact.

**Proof.** Let $l_n$ be compact, then clearly the restriction of $l_n$ to $M(S)$ is compact. Conversely, let $l_n : M(S) \rightarrow LUC(S)^*$ be compact, where $l_n(\mu) = \mu n$ $(\mu \in M(S))$. Let $m \in LUC(S)^*$ with $\|m\| \leq 1$. Since, the linear span of $\delta_x$'s is weak*-dense in $LUC(S)^*$, there is a net $\mu_\alpha = \sum_{i=1}^{k_\alpha} \lambda_{\alpha,i} \delta_{x_{\alpha,i}}$ such that $\mu_\alpha \rightarrow m$ in weak*-topology. By compactness of $l_n$, there is a subnet $(\mu_{\alpha(\beta)})$ such that $(\mu_{\alpha(\beta)} n)$ converges in norm.

Now, we have $mn = \omega^* \cdot \lim \mu_{\alpha(\beta)} n$. Thus $mn = \lim \mu_{\alpha(\beta)} n$ with norm topology. It follows that

$$\{mn \mid \|m\| \leq 1\} \subseteq \{\mu n \mid \mu \in L(S), \|\mu\| \leq 1\}.$$  \hfill (3.6)

Thus, $l_n$ is compact. \hfill $\Box$

**Theorem 3.6.** Let $S$ be a right cancellative semigroup with identity $e$ and $l_n$ a right multiplier on $LUC(S)^*$. Then $l_n$ can be written as a linear combination of four compact right multipliers $l_{n_i}$ $(i = 1, 2, 3, 4)$, $n_i \geq 0$, $n_i \in LUC(S)^*$.

**Proof.** Let $e$ be the identity of $S$. Then $T(m) = T(m\delta_x) = mT(\delta_x)l_T(\delta_x)(m)$. So,$T = l_n$ $(n = T(\delta_x) \in LUC(S)^*)$. Let $n = n\tilde{x} - n\tilde{x} + i(n\tilde{x} - n\tilde{x})$ where $n\tilde{x}, n\tilde{x}$ $(k = 1, 2)$ are Hermitian. So, it suffices to show that $l_{n\tilde{x}}$ and $l_{n\tilde{x}}$ are compact. By Proposition 3.4 it suffices to prove that the restrictions of these operators to $M(S)$ are compact. Now since $l_n$ is compact on $LUC(S)^*$, $\{\delta_x n \mid x \in S\}$ is compact. So $\{\|\delta_x n\| \leq 1\}$ is compact. Since, $\|n^+\| \leq \|n\|$, $\{\delta_x n^+ \mid x \in S\}$ is compact. It follows that $\{\delta_x n^+ \mid x \in S\}$ is compact. Since the linear span of $\delta_x$, $s$ is weak*-dense in $LUC(S)^*$, $\{\mu n^+ \mid \mu \in M(S), \|\mu\| \leq 1\}$ is compact. Therefore, $l_{n^+}$ is compact. This completes the proof. \hfill $\Box$
We denote by \( \beta S \) the space of all multiplicative linear functional on \( LUC(S) \). We have another main theorem.

**Theorem 3.7.** Let \( S \) be a finite topological semigroup. Then there exists \( n \in \beta S \) such that \( l_n \) is compact. Conversely, if \( S \) is a subsemigroup of a topological group with identity, and there exists \( n \in \beta S \) such that \( l_n \) is compact, then \( S \) is finite.

**Proof.** Let \( S \) be finite, then by [2, Corollary 4.1.8], \( AP(S) = C(S) \). Also, by [2, Proposition 4.4.8], \( AP(S) = LUC(S) = RUC(S) \). Therefore, \( LUC(S) = C(S) \). So \( \beta S \) is topologically isomorphic to \( S \). On the other hand, since \( l_S \subseteq S \) is compact, \( l_{\ast}C(S) \) is compact. Hence, \( l_{\ast} \) is compact.

Conversely, let \( l_n \) be compact for some \( n \in \beta S \), by Theorem 3.6, we may assume that \( n \) is positive, then \( T_n(f) = nf \) \( (f \in LUC(S)) \) is compact. Now, let \( F = \text{range} \ T_n \). Clearly \( T_n \) is an algebra homomorphism, since, \( T_n(fg) = n(fg)(x) = \langle n, l_x fg \rangle = \langle n, l_x l_f g \rangle = T_n(l_f)T_n(g) \). Also \( T_n \) preserves conjugation. So, by [8, Theorem 5.3], \( \| T_n f \| \geq \| f \| \) \( (f \in LUC(S)) \). So by open mapping theorem, \( T_n \) is a homeomorphism. Since \( T_n \) is compact, \( F \) is closed. Also, \( \{ T_n f \mid f \in LUC(S), \| T_n f \| \leq 1 \} \subseteq \{ T_n f \mid f \in LUC(S), \| f \| \leq 1 \} \), so \( \{ T_n f \mid f \in LUC(S), \| T_n f \| \leq 1 \} \) is compact. Therefore \( F \) is reflexive. It follows that \( F \) is finite dimensional (see [8, Exercise 2]). Let \( \{ m_1, m_2, \ldots, m_k \} \) be the spectrum of \( F \) and we can assume that \( m_i \) is positive. If we define \( m(f) = (1/k) \sum_{i=1}^k m_i(T_n f) \), then clearly, \( m \geq 0, m(1) = 1 \). Also, since \( S \) is left cancellative, \( l_{\ast} \{ m_1, \ldots, m_k \} = \{ m_1, \ldots, m_k \} \). Therefore, \( \langle m_i, T_n l_{\ast} f \rangle = \langle l_{\ast} m_i, T_n f \rangle = \langle l_{\ast} m_{ij}, T_n f \rangle = \langle m_j, T_n f \rangle \), for some \( 1 \leq j \leq k \). It follows that \( m(l_{\ast} f) = m(f) \), that is, \( m \) is a left-invariant mean on \( LUC(S) \), so by [5, Theorem 3] \( S \) is finite.

**References**


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