We find all complex potentials \( Q \) such that the general Schrödinger operator on \( \mathbb{R}^n \), given by \( L = -\Delta + Q \), where \( \Delta \) is the Laplace differential operator, verifies the well-known Kato's square problem. As an application, we will consider the case where \( Q \in L_{\text{loc}}^1(\Omega) \).

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1. Introduction. Let \( \Omega \subseteq \mathbb{R}^n \), an open set and let's stand in the Hilbert space \( H = L^2(\Omega, C) (= L^2(\Omega)) \). Consider \( Q \), a measurable complex function and let \( \Phi \) and \( \Psi \) be the sesquilinear forms given by,

\[
\Phi(u, v) = \int_{\Omega} \nabla u \nabla v \, dx \quad \forall u, v \in D(\Phi) = H_0^1(\Omega),
\]

\[
\Psi(u, v) = \int_{\Omega} Q u \overline{v} \, dx \quad \forall u, v \in D(\Psi),
\]

where \( D(\Psi) = \{ u \in L^2(\Omega) : |Q|u|^2 \in L^1(\Omega) \} \). Assume that the potential \( Q \) verifies that there exists \( \beta > 0 \) and there exists \( \theta \in (0, \pi/2) \), such that

\[
| \arg(Q - \beta) | \leq \frac{\pi}{2} - \theta.
\]

The sesquilinear forms \( \Phi \) and \( \Psi \) are both closed, densely defined, and sectorial. According to Kato's first representation theorem (see [2]), we can associate to \( \Phi \) and \( \Psi \), \( m \)-sectorial linear operators defined, respectively, by

\[
Au = -\Delta u \quad \text{with} \quad D(A) = \{ u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega) \},
\]

\[
Bu = Qu \quad \text{with} \quad D(B) = \{ u \in L^2(\Omega) : Qu \in L^2(\Omega) \}.
\]

By Schrödinger operator, we mean a partial differential operator on \( \mathbb{R}^n \) of the form

\[
L = A + B; \quad A = -\Delta; \quad B = Q = Q(x),
\]

where \( \Delta \) is the \( n \)-dimensional Laplace operator \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \). The name comes from the form of Schrödinger's equation which, in units with \( h = m = 1 \) reads

\[
i \frac{\partial u}{\partial t} = Lu.
\]

Our aim here is to find all potentials \( Q \) such that

\[
D(L^{1/2}) = D(A^{1/2}) \cap D(B^{1/2}) = D(L^{*1/2}).
\]
For that we use the author results [1] related to the sum of linear operators connected to Kato’s square root problem. The case where $\Omega = \mathbb{R}^n$ will be studied later as a consequence of the general case.

2. Schrödinger operators and Kato’s condition

**Definition 2.1.** A linear operator $C$ is said to verify Kato’s square root problem (or Kato’s condition) if
\[
D(C^{1/2}) = D(Y) = D(C^{*1/2}),
\]
where $Y$ is the sesquilinear form associated to $C$.

**Hypothesis on $Q$.** Suppose that $Q$ is chosen such as,
\[
D(\Phi) \cap D(\Psi) = L^2(\Omega). \tag{2.2}
\]

**Proposition 2.2.** Let $A$ and $B$ be the linear operators given by (1.3). Assume that the potential $Q$ verifies (2.2). Then there exists a unique operator sum $A \oplus B$, which is $m$-sectorial, verifying Kato’s condition and

(i) $A \oplus B = \overline{A + B}$ if $A + B$ is a maximal operator,

(ii) $|\text{Im}( (A \oplus B)u, u) | \leq \text{Re}( (A \oplus B)u, u)$, for all $u \in D(A \oplus B)$.

**Proof.** Assume that $Q$ verifies hypothesis (2.2). So, the sesquilinear form given by, $Y = \Phi + \Psi$, is a closed, sectorial, and densely defined. By Kato’s first representation theorem (see [2]), there exists a unique $m$-sectorial sum operator, $A \oplus B$, associated to $Y$, verifying
\[
Y(u, v) = \langle (A \oplus B)u, v \rangle \quad \forall u \in D(A \oplus B), \; v \in D(\Phi) \cap D(\Psi) \tag{2.3}
\]

since $A$ and $B$ both verify Kato’s condition, according to author’s theorem (see [1, Theorem 2, page 462]), the operator $A \oplus B$ verifies the same condition, that is,
\[
D( (A \oplus B)^{1/2} ) = D(\Phi) \cap D(\Psi) = D( (A \oplus B)^{*1/2} ) \tag{2.4}
\]

and (ii) is satisfied, where $\beta$ is given by (1.2). The operator $A \oplus B$ is defined as
\[
(A \oplus B)u = -\Delta u + Qu, \quad \forall u \in D(A \oplus B),
\]
\[
D(A \oplus B) = \{ u \in H^1_0(\Omega) : Q|u|^2 \in L^1(\Omega), -\Delta u + Qu \in L^2(\Omega) \}. \tag{2.5}
\]

Using also author’s theorem (see [1, Theorem 2, page 462]), it follows that (i) is satisfied.

3. Some applications. Consider the same operators, that is, $A = -\Delta$ and $B = Q$ in $L^2(\Omega)$. Assume that $Q \in L^1_{\text{loc}}(\Omega)$, in this case (2.2) is satisfied. According to Brézis and Kato, the operator $A + B$ is maximal in $L^2(\Omega)$ (then $A \oplus B = A + B$ and Kato’s condition is satisfied) and is given by (2.5).
AN APPLICATION TO KATO’S SQUARE ROOT PROBLEM

**Case** $\Omega = \mathbb{R}^n$. We always assume $Q \in L^1_{\text{loc}}(\mathbb{R}^n)$, it follows that

\begin{align*}
Au &= -\Delta u \quad \text{with} \quad D(A) = H^2(\mathbb{R}^n); \quad D(A^{1/2}) = H^1(\mathbb{R}^n), \\
Bu &= Qu \quad \text{with} \quad D(B^{1/2}) = \{u \in L^2(\mathbb{R}^n) : Q|u|^2 \in L^1(\mathbb{R}^n)\},
\end{align*}

and $D(A^{1/2}) \cap D(B^{1/2})$ is dense in $L^2(\mathbb{R}^n)$ because,

\begin{equation}
C_0^\infty(\mathbb{R}^n) \subseteq D(\Phi) \cap D(\Psi). \tag{3.2}
\end{equation}

In conclusion, Kato’s condition is satisfied by $A + B$, that is,

\begin{equation}
D\left(\sqrt{A+B}\right) = D(\sqrt{A}) \cap D(\sqrt{B}) = D\left(\sqrt{A+B^*}\right). \tag{3.3}
\end{equation}

For example when $n = 1$, then

\begin{equation}
D\left(\sqrt{A+B}\right) = H^1(\mathbb{R}) = D\left(\sqrt{A+B^*}\right). \tag{3.4}
\end{equation}

**Remark 3.1.** Condition (2.2) could be weakened as

\begin{equation}
\overline{D(A)} \cap \overline{D(B)} = L^2(\Omega). \tag{3.5}
\end{equation}

But in general the algebraic sum of two operators is not always defined (because this concept is not well adapted to problems arising in mathematical analysis).

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**References**


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