Every affine structure on Lie algebra $\mathfrak{g}$ defines a representation of $\mathfrak{g}$ in $\text{aff}(\mathbb{R}^n)$. If $\mathfrak{g}$ is a nilpotent Lie algebra provided with a complete affine structure then the corresponding representation is nilpotent. We describe noncomplete affine structures on the filiform Lie algebra $L_n$. As a consequence we give a nonnilpotent faithful linear representation of the 3-dimensional Heisenberg algebra.

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1. Affine structure on a nilpotent Lie algebra

1.1. Affine structure on nilpotent Lie algebras

**Definition 1.1.** Let $\mathfrak{g}$ be an $n$-dimensional Lie algebra over $\mathbb{R}$. An affine structure is given by a bilinear mapping

$$\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

$$\nabla(X,Y) - \nabla(Y,X) = [X,Y],$$

$$\nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) = \nabla([X,Y],Z),$$

for all $X,Y,Z \in \mathfrak{g}$.

If $\mathfrak{g}$ is provided with an affine structure, then the corresponding connected Lie group $G$ is an affine manifold such that every left translation is an affine isomorphism of $G$. In this case, the operator $\nabla$ is nothing but the connection operator of the affine connection on $G$.

Let $\mathfrak{g}$ be a Lie algebra with an affine structure $\nabla$. Then the mapping

$$f : \mathfrak{g} \to \text{End}(\mathfrak{g}),$$

defined by

$$f(X)(Y) = \nabla(X,Y),$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y) - f(Y)(X) = [X,Y].$$
Remark 1.2. The adjoint representation $\tilde{f}$ of $g$ satisfies
\[ \tilde{f}(X)(Y) - \tilde{f}(Y)(X) = 2[X, Y] \] (1.6)
and cannot correspond to an affine structure.

1.2. Classical examples of affine structures. (i) Let $g$ be the $n$-dimensional abelian Lie algebra. Then the representation
\[ f : g \to \text{End}(g), \quad X \mapsto f(X) = 0 \] (1.7)
defines an affine structure.

(ii) Let $g$ be a $2p$-dimensional Lie algebra endowed with a symplectic form $\theta \in \Lambda^2 g^*$ such that $d\theta = 0$ (1.8) with
\[ d\theta(X, Y, Z) = \theta([X, Y, Z]) + \theta([Z, X, Y]) + \theta([X, Y, Z]). \] (1.9)
For every $X \in g$ we can define a unique endomorphism $\nabla_X$ by
\[ \theta(\text{ad}_X(Y), Z) = -\theta(Y, \nabla_X(Z)). \] (1.10)
Then $\nabla(X, Y) = \nabla_X(Y)$ is an affine structure on $g$.

(iii) Following the work of Benoist [1] and Burde [2, 3, 4], we know that there exists a nilpotent Lie algebra without affine structures.

1.3. Faithful representations associated to an affine structure. Let $\nabla$ be an affine structure on an $n$-dimensional Lie algebra $g$. We consider the $(n+1)$-dimensional linear representation given by
\[ \rho : g \to \text{End}(g \oplus \mathbb{R}) \] (1.11)
given by
\[ \rho(X) : (Y, t) \mapsto (\nabla(X, Y) + tX, 0). \] (1.12)
It is easy to verify that $\rho$ is a faithful representation of dimension $n+1$.

We can note that this representation gives also an affine representation of $g$
\[ \psi : g \to \text{aff}(\mathbb{R}^n), \quad X \mapsto \begin{pmatrix} A(X) & X \\ 0 & 0 \end{pmatrix}, \] (1.13)
where $A(X)$ is the matrix of the endomorphisms $\nabla_X : Y \to \nabla(X, Y)$ in a given basis.

Definition 1.3. We say that the representation $\rho$ is nilpotent if the endomorphisms $\rho(X)$ are nilpotent for every $X$ in $g$.

Proposition 1.4. Suppose that $g$ is a complex non-abelian indecomposable nilpotent Lie algebra and let $\rho$ be a faithful representation of $g$. Then there exists a faithful nilpotent representation of the same dimension.
Proof. Consider the \( g \)-module \( M \) associated to \( \rho \). Then, as \( g \) is nilpotent, \( M \) can be decomposed as

\[
M = \bigoplus_{i=1}^{k} M_{\lambda_i},
\]

where \( M_{\lambda_i} \) is a \( g \)-submodule, and the \( \lambda_i \) are linear forms on \( g \). For all \( X \in g \), the restriction of \( \rho(X) \) to \( M_i \) is in the following form:

\[
\begin{pmatrix}
\lambda_i(X) & * & \cdots & * \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & \lambda_i(X)
\end{pmatrix}.
\]

(1.15)

Let \( \mathbb{K}_{\lambda_i} \) be the one-dimensional \( g \)-module defined by

\[
\mu : X \in g \mapsto \mu(X) \in \text{End } \mathbb{K}
\]

with

\[
\mu(X)(a) = \rho(X)(a) = \lambda_i(X)a.
\]

(1.17)

The tensor product \( M_{\lambda_i} \otimes \mathbb{K}_{\lambda_i} \) is the \( g \)-module associated to

\[
X \cdot (Y \otimes a) = \rho(X)(Y) \otimes a - Y \otimes \lambda_i(X)a.
\]

(1.18)

Then \( \tilde{M} = \bigoplus (M_{\lambda_i} \otimes \mathbb{K}_{-\lambda_i}) \) is a nilpotent \( g \)-module. We prove that \( \tilde{M} \) is faithful. Recall that a representation \( \rho \) of \( g \) is faithful if and only if \( \rho(Z) \neq 0 \) for every \( Z \neq 0 \in Z(g) \). Consider \( X \neq 0 \in Z(g) \). If \( \tilde{\rho}(X) = 0 \), then \( \rho(X) \) is a diagonal endomorphism. By hypothesis \( g \neq Z(g) \) and there is \( i \geq 1 \) such that \( X \in \mathfrak{c}^i(g) \), we have

\[
X = \sum_{j} a_j [Y_j, Z_j]
\]

(1.19)

with \( Y_j \in \mathfrak{c}^{i-1}(g) \) and \( Z_j \in g \). The endomorphisms \( \rho(Y_j) \rho(Z_j) - \rho(Z_j) \rho(Y_j) \) are nilpotent and the eigenvalues of \( \rho(X) \) are 0. Thus \( \rho(X) = 0 \) and \( \rho \) is not faithful. Then \( \tilde{\rho}(X) \neq 0 \) and \( \tilde{\rho} \) is a faithful representation.

2. Affine structures on Lie algebra of maximal class

2.1. Definition

Definition 2.1. An \( n \)-dimensional nilpotent Lie algebra \( g \) is called of maximal class if the smallest \( k \) such that \( \mathfrak{c}^k g = \{0\} \) is equal to \( n - 1 \).

In this case the descending sequence is

\[
g \supset \mathfrak{c}^1 g \supset \cdots \supset \mathfrak{c}^{n-2} g \supset \{0\} = \mathfrak{c}^{n-1} g
\]

(2.1)

and we have

\[
\dim \mathfrak{c}^1 g = n - 2,
\]

\[
\dim \mathfrak{c}^i g = n - i - 1, \quad \text{for } i = 1, \ldots, n - 1.
\]

(2.2)
**Example 2.2.** The $n$-dimensional nilpotent Lie algebra $L_n$ defined by

$$[X_1, X_i] = X_{i+1} \quad \text{for } i \in \{2, \ldots, n-1\}$$

(2.3)

is of maximal class.

We can note that any Lie algebra of maximal class is a linear deformation of $L_n$ [5].

2.2. **On non-nilpotent affine structure.** Let $g$ be an $n$-dimensional Lie algebra of maximal class provided with an affine structure $\nabla$. Let $\rho$ be the $(n+1)$-dimensional faithful representation associated to $\nabla$ and we note that $M = g \bigoplus \mathbb{C}$ is the corresponding complex $g$-module. As $g$ is of maximal class, its decomposition has one of the following forms

$$M = M_0, \quad M \text{ is irreducible},$$

(2.4)

or

$$M = M_0 \bigoplus M_\lambda, \quad \lambda \neq 0.$$  

(2.5)

For a general faithful representation, we call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the case of maximal class we have $c(\rho) = (n+1)$ or $(n, 1)$ or $(n-1, 1, 1)$ or $(n-1, 2)$. In fact, the maximal class of $g$ implies that there exists an irreducible submodule of dimension greater than or equal to $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, \ldots, c_p, 1)$ (see [5]) then for every faithful representation $\rho$ we have $c(\rho) = (d_1, \ldots, d_q)$ with $d_1 \geq c_1$.

**Theorem 2.3.** Let $g$ be the Lie algebra of the maximal class $L_n$. Then there are faithful $g$-modules which are not nilpotent.

**Proof.** Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, \ldots, X_n\}$ is a basis of $g$

$$\rho(X_1) = \begin{pmatrix}
a & a & 0 & \cdots & \cdots & 0 & 1 \\
a & a & 0 & & & & \\
0 & 0 & 0 & & & & \\
\vdots & \ddots & \frac{1}{2} & & & & \\
\vdots & \ddots & \ddots & \ddots & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & & \\
0 & 0 & \cdots & \cdots & \frac{i-3}{i-2} & \ddots & \\
0 & 0 & \cdots & \cdots & \cdots & \ddots & \\
\alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
and for \( j \geq 3 \) the endomorphisms \( \rho(X_j) \) satisfy

\[
\rho(X_j)(e_1) = -\frac{1}{j-1}e_{j+1},
\]

\[
\rho(X_j)(e_2) = \frac{1}{j-1}e_{j+1},
\]

\[
\rho(X_j)(e_3) = \frac{1}{j(j-1)}e_{j+2},
\]

\[
\vdots
\]

\[
\rho(X_j)(e_{i-j+1}) = \frac{(j-2)!(i-j-1)!}{(i-2)!}e_i, \quad i = j-2, \ldots, n,
\]

\[
\rho(X_j)(e_{i-j+1}) = 0, \quad i = n+1, \ldots, n+j-1,
\]

\[
\rho(X_j)(e_{n+1}) = e_j,
\]

where \( \{e_1, \ldots, e_n, e_{n+1}\} \) is the basis given by \( e_i = (X_i, 0) \) and \( e_{n+1} = (0, 1) \). We easily verify that these matrices describe a nonnilpotent faithful representation. \( \square \)

2.3. **Noncomplete affine structure on** \( L_n \). The previous representation is associated to an affine structure on the Lie algebra \( L_n \) given by

\[
\nabla(X_i, Y) = \rho(X_i)(Y, 0),
\]

where \( L_n \) is identified to the \( n \)-dimensional first factor of the \( (n+1) \)-dimensional faithful module. This affine structure is complete if and only if the endomorphisms \( R_X \in \text{End}(g) \) defined by

\[
R_X(Y) = \nabla(Y, X)
\]
are nilpotent for all $X \in \mathfrak{g}$ (see [6]). But the matrix of $R_{X_1}$ has the form

$$
\begin{pmatrix}
    a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\
    a & a & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 1 \\
    \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
    0 & 0 & \vdots & \ddots & \frac{1}{j-1} & \vdots & \vdots & \vdots \\
    \alpha & \beta & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0
\end{pmatrix}.
$$

(2.10)

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved the following proposition.

**Proposition 2.4.** There exist affine structures on the Lie algebra of maximal class $L_n$ which are noncomplete.

**Remark 2.5.** The most simple example is on dim3 and concerns the Heisenberg algebra. We find a nonnilpotent faithful representation associated to the noncomplete affine structure given by

$$
\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta - 1 & \alpha + 1 & 0 \end{pmatrix}, \quad \nabla_{X_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

(2.11)

where $X_1$, $X_2$, and $X_3$ are a basis of $H_3$ satisfying $[X_1, X_2] = X_3$ and $\nabla_{X_i}$ the endomorphisms of $\mathfrak{g}$ given by

$$
\nabla_{X_i}(X_j) = \nabla(X_i, X_j).
$$

(2.12)

The affine representation is written as

$$
\begin{pmatrix}
    a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_1 \\
    a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_2 \\
    \alpha x_1 + (\beta - 1)x_2 & \beta x_1 + (\alpha + 1)x_2 & 0 & x_3 \\
    0 & 0 & 0 & 0
\end{pmatrix}.
$$

(2.13)

**References**


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