ON $\theta$-PRECONTINUOUS FUNCTIONS

TAKASHI NOIRI

(Received 16 January 2001)

ABSTRACT. We introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous (or almost weakly continuous) functions and contains the class of almost precontinuous functions. It is shown that the $\theta$-precontinuous image of a $p$-closed space is quasi $H$-closed.

2000 Mathematics Subject Classification. 54C08.

1. Introduction. A subset $A$ of a topological space $X$ is said to be preopen [14] or nearly open [26] if $A \subset \text{Int}(\text{Cl}(A))$. A function $f : X \to Y$ is called precontinuous [14] if the preimage $f^{-1}(V)$ of each open set $V$ of $Y$ is preopen in $X$. Precontinuity was called near continuity by Pták [26] and also called almost continuity by Frolík [9] and Husain [10]. In 1985, Janković [12] introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri [23] introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. Paul and Bhattacharyya [21] called weakly precontinuous functions quasi precontinuous and obtained the further properties of quasi precontinuity. Recently, Nasef and Noiri [16] have introduced and investigated the notion of almost precontinuity. Quite recently, Jafari and Noiri [11] investigated the further properties of almost precontinuous functions.

In this paper, we introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous functions and contains the class of almost precontinuous functions. We obtain basic properties of $\theta$-precontinuous functions. It is shown in the last section that the $\theta$-precontinuous images of $p$-closed (resp., $\beta$-connected) spaces are quasi $H$-closed (resp., semi-connected).

2. Preliminaries. Throughout, by $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) we denote topological spaces. Let $S$ be a subset of $X$. We denote the interior and the closure of $S$ by $\text{Int}(S)$ and $\text{Cl}(S)$, respectively. A subset $S$ is said to be preopen [14] (resp., semi-open [13], $\alpha$-open [17]) if $S \subset \text{Int}(\text{Cl}(S))$ (resp., $S \subset \text{Cl}(\text{Int}(S))$). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing $S$ is called the preclosure [8] of $S$ and is denoted by $\text{pCl}(S)$. The preinterior of $S$ is defined by the union of all preopen sets contained in $S$ and is denoted by $\text{pInt}(S)$. The family of all preopen sets of $X$ is denoted by $\text{PO}(X)$. We set $\text{PO}(X,x) = \{U : x \in U \text{ and } U \in \text{PO}(X)\}$. A point $x$ of $X$ is called a $\theta$-cluster point of $S$ if $\text{Cl}(U) \cap S \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $S$ is called the $\theta$-closure of $S$ and is denoted by $\text{Clop}(S)$. A subset $S$ is said to be $\theta$-closed [27] if $S = \text{Clop}(S)$. The complement of a $\theta$-closed set is said to be $\theta$-open. A point $x$ of $X$
is called a pre $\theta$-cluster point of $S$ if \( p\text{Cl}(U) \cap S \neq \emptyset \) for every preopen set $U$ of $X$ containing $x$. The set of all pre-$\theta$-cluster points of $S$ is called the pre $\theta$-closure of $S$ and is denoted by $p\text{Cl}_\theta(S)$. A subset $S$ is said to be pre $\theta$-closed [20] if $S = p\text{Cl}_\theta(S)$.

The complement of a pre $\theta$-open set is called a pre $\theta$-closed set.

**Definition 2.1.** A function $f : X \rightarrow Y$ is said to be precontinuous [14] (resp., almost precontinuous [16], weakly precontinuous [23] or quasi precontinuous [21]) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(U) \subset V$ (resp., $f(U) \subset \text{Int}(\text{Cl}(V))$, $f(U) \subset \text{Cl}(V)$).

**Definition 2.2.** A function $f : X \rightarrow Y$ is said to be almost weakly continuous [12] if $f^{-1}(V) \subset \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V))))$ for every open set $V$ of $Y$.

**Definition 2.3.** A function $f : X \rightarrow Y$ is said to be strongly $\theta$-precontinuous [19] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset V$.

**Definition 2.4.** A function $f : X \rightarrow Y$ is said to be $\theta$-precontinuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$.

**Remark 2.5.** By the above definitions and Theorem 3.3 below, we have the following implications and none of these implications is reversible by [19, Example 2.2], [11, Example 2.9], and Examples 2.6 and 5.11 below.

$$\text{strongly } \theta\text{-precontinuous} \Rightarrow \text{precontinuous} \Rightarrow \text{almost precontinuous}$$

$$\Rightarrow \theta\text{-precontinuous} \Rightarrow \text{weakly precontinuous}.$$

**Example 2.6.** This example is due to Arya and Deb [4]. Let $X$ be the set of all real numbers. The topology $\tau$ on $X$ is the cocountable topology. Let $Y = \{a, b, c\}$, $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. We define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(x) = a$ if $x$ is rational; $f(x) = b$ if $x$ is irrational. Then $f$ is a $\theta$-precontinuous function which is not almost precontinuous.

3. Characterizations

**Theorem 3.1.** For a function $f : X \rightarrow Y$ the following properties are equivalent:

1. $f$ is $\theta$-precontinuous;
2. $p\text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$ for every subset $B$ of $Y$;
3. $f(p\text{Cl}_\theta(A)) \subset \text{Cl}_\theta(f(A))$ for every subset $A$ of $X$.

**Proof.** (1)$\Rightarrow$(2). Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(\text{Cl}_\theta(B))$. Then $f(x) \notin \text{Cl}_\theta(B)$ and there exists an open set $V$ containing $f(x)$ such that $\text{Cl}(V) \cap B = \emptyset$. Since $f$ is $\theta$-p.c., there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$. Therefore, we have $f(p\text{Cl}(U)) \cap B = \emptyset$ and $p\text{Cl}(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin p\text{Cl}_\theta(f^{-1}(B))$. Thus, we obtain $p\text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$. 

2.$\Rightarrow$(1). Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(\text{Cl}_\theta(B))$. Then $f(x) \notin \text{Cl}_\theta(B)$ and there exists an open set $V$ containing $f(x)$ such that $\text{Cl}(V) \cap B = \emptyset$. Since $f$ is $\theta$-p.c., there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$. Therefore, we have $f(p\text{Cl}(U)) \cap B = \emptyset$ and $p\text{Cl}(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin p\text{Cl}_\theta(f^{-1}(B))$. Thus, we obtain $p\text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$. 


(2)⇒(3). Let A be any subset of X. Then we have \( p\text{Cl}_\theta(A) \subset p\text{Cl}_\theta(f^{-1}(f(A))) \subset f^{-1}(\text{Cl}_\theta(f(A))) \) and hence \( f(p\text{Cl}_\theta(A)) \subset \text{Cl}_\theta(f(A)). \)

(3)⇒(2). Let B be a subset of Y. We have \( f(p\text{Cl}_\theta(f^{-1}(B))) \subset \text{Cl}_\theta(f(f^{-1}(B))) \subset \text{Cl}_\theta(B) \) and hence \( p\text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B)). \)

(2)⇒(1). Let \( x \in X \) and V be an open set of Y containing \( f(x). \) Then we have \( \text{Cl}(V) \cap (Y - \text{Cl}(V)) = \emptyset \) and \( f(x) \notin \text{Cl}_\theta(Y - \text{Cl}(V)). \) Hence, \( x \notin f^{-1}(\text{Cl}_\theta(Y - \text{Cl}(V))) \) and \( x \notin p\text{Cl}_\theta(f^{-1}(Y - \text{Cl}(V))). \) There exists \( U \in \text{PO}(X,x) \) such that \( p\text{Cl}(U) \cap f^{-1}(Y - \text{Cl}(V)) = \emptyset; \) hence \( f(p\text{Cl}(U)) \subset \text{Cl}(V). \) Therefore, \( f \) is \( \theta.p.c. \)

**Theorem 3.2.** For a function \( f : X \to Y \) the following properties are equivalent:

1. \( f \) is \( \theta \)-precontinuous;
2. \( f^{-1}(V) \subset \text{plnt}_\theta(f^{-1}(\text{Cl}(V))) \) for every open set \( V \) of \( Y; \)
3. \( p\text{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V)) \) for every open set \( V \) of \( Y. \)

**Proof.** (1)⇒(2). Suppose that \( V \) is any open set of \( Y \) and \( x \in f^{-1}(V). \) Then \( f(x) \in V \) and there exists \( U \in \text{PO}(X,x) \) such that \( f(p\text{Cl}(U)) \subset \text{Cl}(V). \) Therefore, \( x \in U \subset p\text{Cl}(U) \subset f^{-1}(\text{Cl}(V)). \) This shows that \( x \in \text{plnt}_\theta(f^{-1}(\text{Cl}(V))). \) Therefore, we obtain \( f^{-1}(V) \subset \text{plnt}_\theta(f^{-1}(V)). \)

(2)⇒(3). Suppose that \( V \) is any open set of \( Y \) and \( x \notin f^{-1}(\text{Cl}(V)). \) Then \( f(x) \notin \text{Cl}(V) \) and there exists an open set \( W \) containing \( f(x) \) such that \( W \cap V = \emptyset; \) hence \( \text{Cl}(W) \cap V = \emptyset. \) Therefore, we have \( f^{-1}(\text{Cl}(W)) \cap f^{-1}(V) = \emptyset. \) Since \( x \in f^{-1}(W), \) by (2) \( x \in \text{plnt}_\theta(f^{-1}(\text{Cl}(W))). \) There exists \( U \in \text{PO}(X,x) \) such that \( p\text{Cl}(U) \subset f^{-1}(\text{Cl}(W)). \) Thus we have \( p\text{Cl}(U) \cap f^{-1}(V) = \emptyset \) and hence \( x \notin p\text{Cl}_\theta(f^{-1}(V)). \) This shows that \( p\text{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V)). \)

(3)⇒(1). Suppose that \( x \in X \) and \( V \) is any open set of \( Y \) containing \( f(x). \) Then \( V \cap (Y - \text{Cl}(V)) = \emptyset \) and \( f(x) \notin \text{Cl}(Y - \text{Cl}(V)). \) Therefore, \( x \notin f^{-1}(\text{Cl}(Y - \text{Cl}(V))) \) and by (3) \( x \notin p\text{Cl}_\theta(f^{-1}(Y - \text{Cl}(V))). \) There exists \( U \in \text{PO}(X,x) \) such that \( p\text{Cl}(U) \cap f^{-1}(Y - \text{Cl}(V)) = \emptyset. \) Therefore, we obtain \( f(p\text{Cl}(U)) \subset \text{Cl}(V). \) This shows that \( f \) is \( \theta.p.c. \)

**Theorem 3.3.** For a function \( f : X \to Y \) the following properties hold:

1. If \( f \) is almost precontinuous, then it is \( \theta \)-precontinuous;
2. If \( f \) is \( \theta \)-precontinuous, then it is weakly precontinuous.

**Proof.** Statement (2) is obvious. We will show statement (1). Suppose that \( x \in X \) and V is any open set of Y containing \( f(x). \) Since \( f \) is almost precontinuous, \( f^{-1}(\text{Int}(\text{Cl}(V))) \) is preopen and \( f^{-1}(\text{Cl}(V)) \) is preclosed in \( X \) by [16, Theorem 3.1]. Now, set \( U = f^{-1}(\text{Int}(\text{Cl}(V))). \) Then we have \( U \in \text{PO}(X,x) \) and \( p\text{Cl}(U) \subset f^{-1}(\text{Cl}(V)). \) Therefore, we obtain \( f(p\text{Cl}(U)) \subset \text{Cl}(V). \) This shows that \( f \) is \( \theta.p.c. \)

**Corollary 3.4.** Let \( Y \) be a regular space. Then, for a function \( f : X \to Y \) the following properties are equivalent:

1. \( f \) is strongly \( \theta \)-precontinuous;
2. \( f \) is precontinuous;
3. \( f \) is almost precontinuous;
4. \( f \) is \( \theta \)-precontinuous;
5. \( f \) is weakly precontinuous.

**Proof.** This is an immediate consequence of [19, Theorem 3.2].
**Definition 3.5.** A topological space $X$ is said to be *pre-regular* [20] if for each preclosed set $F$ and each point $x \in X - F$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

**Lemma 3.6 (see [20]).** A topological space $X$ is pre-regular if and only if for each $U \in \text{PO}(X)$ and each point $x \in U$, there exists $V \in \text{PO}(X,x)$ such that $x \in V \subset \text{pCl}(V) \subset U$.

**Theorem 3.7.** Let $X$ be a pre-regular space. Then $f : X \to Y$ is \(\theta\).p.c. if and only if it is weakly precontinuous.

**Proof.** Suppose that $f$ is weakly precontinuous. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then, there exists $U \in \text{PO}(X,x)$ such that $f(U) \subset \text{Cl}(V)$.

Since $X$ is pre-regular, there exists $U_* \in \text{PO}(X,x)$ such that $x \in U_* \subset \text{pCl}(U_*) \subset U$. Therefore, we obtain $f(p\text{Cl}(U_*)) \subset \text{Cl}(V)$. This shows that $f$ is \(\theta\).p.c. \hfill \Box

**Theorem 3.8.** Let $f : X \to Y$ be a function and $g : X \times Y \to Y$ the graph function of $f$ defined by $g(x) = (x, f(x))$ for each $x \in X$. Then $g$ is \(\theta\).p.c. if and only if $f$ is \(\theta\).p.c.

**Proof.**

**Necessity.** Suppose that $g$ is \(\theta\).p.c. Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since $g$ is \(\theta\).p.c., there exists $U \in \text{PO}(X,x)$ such that $g(p\text{Cl}(U)) \subset \text{Cl}(X \times V)$. It follows that $\text{Cl}(X \times V) = X \times \text{Cl}(V)$. Therefore, we obtain $f(p\text{Cl}(U)) \subset \text{Cl}(V)$. This shows that $f$ is \(\theta\).p.c.

**Sufficiency.** Let $x \in X$ and $W$ be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subset W$. Since $f$ is \(\theta\).p.c., there exists $U_2 \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U_2)) \subset \text{Cl}(V)$. Let $U = U_1 \cap U_2$, then $U \in \text{PO}(X,x)$. Therefore, we obtain $g(p\text{Cl}(U)) \subset \text{Cl}(U_1) \times f(p\text{Cl}(U_2)) \subset \text{Cl}(U_1) \times \text{Cl}(V) \subset \text{Cl}(W)$. This shows that $g$ is \(\theta\).p.c. \hfill \Box

4. Some properties

**Lemma 4.1 (see [15]).** Let $A$ and $X_0$ be subsets of a space $X$.

1. If $A \in \text{PO}(X)$ and $X_0$ is semi-open in $X$, then $A \cap X_0 \in \text{PO}(X_0)$.
2. If $A \in \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $A \in \text{PO}(X)$.

**Lemma 4.2 (see [7]).** Let $A$ and $X_0$ be subsets of a space $X$ such that $A \subset X_0 \subset X$. Let $p\text{Cl}_{X_0}(A)$ denote the preclosure of $A$ in the subspace $X_0$.

1. If $X_0$ is semi-open in $X$, then $p\text{Cl}_{X_0}(A) \subset p\text{Cl}(A)$.
2. If $A \in \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $p\text{Cl}(A) \subset p\text{Cl}_{X_0}(A)$.

**Theorem 4.3.** If $f : X \to Y$ is \(\theta\).p.c. and $X_0$ is a semi-open subset of $X$, then the restriction $f/X_0 : X_0 \to Y$ is \(\theta\).p.c.

**Proof.** For any $x \in X_0$ and any open neighborhood $V$ of $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$ since $f$ is \(\theta\).p.c. Put $U_0 = U \cap X_0$, then by Lemmas 4.1 and 4.2 $U_0 \in \text{PO}(X_0,x)$ and $p\text{Cl}_{X_0}(U_0) \subset p\text{Cl}(U_0)$. Therefore, we obtain

$$
(f/X_0)(p\text{Cl}_{X_0}(U_0)) = f(p\text{Cl}_{X_0}(U_0)) \subset f(p\text{Cl}(U_0)) \subset f(p\text{Cl}(U)) \subset \text{Cl}(V). \quad (4.1)
$$

This shows that $f/X_0$ is \(\theta\).p.c. \hfill \Box
THEOREM 4.4. A function \( f : X \to Y \) is \( \theta.p.c. \) if for each \( x \in X \) there exists \( X_0 \in \text{PO}(X,x) \) such that the restriction \( f/X_0 : X_0 \to Y \) is \( \theta.p.c. \).

**Proof.** Let \( x \in X \) and \( V \) be any open neighborhood of \( f(x) \). There exists \( X_0 \in \text{PO}(X,x) \) such that \( f/X_0 : X_0 \to Y \) is \( \theta.p.c. \). Thus, there exists \( U \in \text{PO}(X,x) \) such that \( (f/X_0)(\text{pCl}_{X_0}(U)) \subset \text{Cl}(V) \). By Lemmas 4.1 and 4.2, \( U \in \text{PO}(X,x) \) and \( \text{pCl}(U) \subset \text{pCl}_{X_0}(U) \). Hence, we have \( f(\text{pCl}(U)) = (f/X_0)(\text{pCl}(U)) \subset (f/X_0)(\text{pCl}_{X_0}(U)) \subset \text{Cl}(V) \). This shows that \( f \) is \( \theta.p.c. \).

COROLLARY 4.5. Let \( \{U_\lambda : \lambda \in \Lambda \} \) be an \( \alpha \)-open cover of a topological space \( X \). A function \( f : X \to Y \) is \( \theta.p.c. \) if and only if the restriction \( f/U_\lambda : U_\lambda \to Y \) is \( \theta.p.c. \) for each \( \lambda \in \Lambda \).

**Proof.** This is an immediate consequence of Theorems 4.3 and 4.4.

Let \( \{X_\alpha : \alpha \in \mathcal{A} \} \) be a family of topological spaces, \( A_\alpha \) a nonempty subset of \( X_\alpha \) for each \( \alpha \in \mathcal{A} \) and \( X = \Pi \{X_\alpha : \alpha \in \mathcal{A} \} \) denote the product space, where \( \mathcal{A} \) is nonempty.

**Lemma 4.6** (see [8]). Let \( n \) be a positive integer and \( A = \Pi_{j=1}^n A_\alpha_j \times \Pi_{\alpha \neq \alpha_j} X_\alpha \).

1. \( A \in \text{PO}(X) \) if and only if \( A_\alpha_j \in \text{PO}(X_\alpha_j) \) for each \( j = 1, 2, \ldots, n \).
2. \( \text{pCl}(\Pi_{\alpha \in \mathcal{A}} A_\alpha) \subset \Pi_{\alpha \in \mathcal{A}} \text{pCl}(A_\alpha) \).

**Theorem 4.7.** If a function \( f_\alpha : X_\alpha \to Y_\alpha \) is \( \theta.p.c. \) for each \( \alpha \in \mathcal{A} \). Then the product function \( f : \Pi X_\alpha \to \Pi Y_\alpha \) defined by \( f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\} \) for each \( x = \{x_\alpha\} \), is \( \theta.p.c. \).

**Proof.** Let \( x = \{x_\alpha\} \in \Pi X_\alpha \) and \( W \) be any open set of \( \Pi Y_\alpha \) containing \( f(x) \). Then, there exists an open set \( V_{\alpha_j} \) of \( Y_{\alpha_j} \) such that

\[
f(x) = \{f_\alpha(x_\alpha)\} \in \Pi_{j=1}^n V_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} Y_\alpha \subset W.
\]

(4.2)

Since \( f_{\alpha} \) is \( \theta.p.c. \) for each \( \alpha \), there exists \( U_{\alpha_j} \in \text{PO}(X_\alpha_j, x_\alpha_j) \) such that \( f_\alpha_j(\text{pCl}(U_{\alpha_j})) \subset \text{Cl}(V_{\alpha_j}) \) for \( j = 1, 2, \ldots, n \). Now, put \( U = \Pi_{j=1}^n U_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} X_\alpha \). Then, it follows from Lemma 4.6 that \( U \in \text{PO}(\Pi X_\alpha, x) \). Moreover, we have

\[
f(\text{pCl}(U)) \subset f(\Pi_{j=1}^n \text{pCl}(U_{\alpha_j}) \times \Pi_{\alpha \neq \alpha_j} X_\alpha)
\]

\[
\subset \Pi_{j=1}^n f_\alpha_j(\text{pCl}(U_{\alpha_j})) \times \Pi_{\alpha \neq \alpha_j} Y_\alpha
\]

(4.3)

This shows that \( f \) is \( \theta.p.c. \).

\section*{5. Preservation property}

**Definition 5.1.** A topological space \( X \) is said to be

1. \( p \)-closed [7] (resp., \( p \)-Lindelöf) if every cover of \( X \) by preopen sets has a finite (resp., countable) subfamily whose preclosures cover \( X \),
2. countably \( p \)-closed if every countable cover of \( X \) by preopen sets has a finite subfamily whose preclosures cover \( X \);,
3. quasi \( H \)-closed [25] (resp., almost Lindelöf [6]) if every cover of \( X \) by open sets has a finite (resp., countable) subfamily whose closures cover \( X \),
4. lightly compact [5] if every countable cover of \( X \) by open sets has a finite subfamily whose closures cover \( X \).
**Definition 5.2.** A subset $K$ of a space $X$ is said to be

1. $p$-closed relative to $X$ [7] if for every cover $\{V_\alpha : \alpha \in A\}$ of $K$ by preopen sets of $X$, there exists a finite subset $A_*$ of $A$ such that $K \subset \bigcup \{p\text{Cl}(V_\alpha) : \alpha \in A_*\}$,
2. quasi $H$-closed relative to $X$ [25] if for every cover $\{V_\alpha : \alpha \in A\}$ of $K$ by open sets of $X$, there exists a finite subset $A_*$ of $A$ such that $K \subset \bigcup \{\text{Cl}(V_\alpha) : \alpha \in A_*\}$.

**Theorem 5.3.** If $f : X \rightarrow Y$ is a $\theta.p.c.$ function and $K$ is $p$-closed relative to $X$, then $f(K)$ is quasi $H$-closed relative to $Y$.

**Proof.** Suppose that $f : X \rightarrow Y$ is $\theta.p.c.$ and $K$ is $p$-closed relative to $X$. Let $\{V_\alpha : \alpha \in A\}$ be a cover of $f(K)$ by open sets of $Y$. For each point $x \in K$, there exists $\alpha(x) \in A$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\theta.p.c.$, there exists $U_x \in \text{PO}(X, x)$ such that $f(p\text{Cl}(U_x)) \subset \text{Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of $K$ by preopen sets of $X$ and hence there exists a finite subset $K_*$ of $K$ such that $K \subset \bigcup_{x \in K_*} p\text{Cl}(U_x)$. Therefore, we obtain $f(K) \subset \bigcup_{x \in K_*} \text{Cl}(V_{\alpha(x)})$. This shows that $f(K)$ is quasi $H$-closed relative to $Y$. 

**Corollary 5.4.** Let $f : X \rightarrow Y$ be a $\theta.p.c.$ surjection. Then, the following properties hold:

1. If $X$ is $p$-closed, then $Y$ is quasi $H$-closed.
2. If $X$ is $p$-Lindelöf, then $Y$ is almost Lindelöf.
3. If $X$ is countably $p$-closed, then $Y$ is lightly compact.

A subset $S$ of a topological space $X$ is said to be $\beta$-open [1] or semipreopen [3] if $S \subset \text{Cl}(\text{Int}(\text{Cl}(S)))$. It is well known that $\alpha$-openness implies both preopenness and semi-openness which imply $\beta$-openness. The complement of a semipreopen set is said to be semipreclosed [3]. The intersection of all semipreclosed sets of $X$ containing a subset $S$ is the semipreclosure of $S$ and is denoted by $sp\text{Cl}(S)$ [3].

**Definition 5.5.** A topological space $X$ is said to be

1. $\beta$-connected [24] or semipreconnected [2] if $X$ cannot be expressed as the union of two nonempty disjoint $\beta$-open sets,
2. semi-connected [22] if $X$ cannot be expressed as the union of two nonempty disjoint semi-open sets.

**Remark 5.6.** We have the following implications:

$$\beta\text{-connected} \Rightarrow \text{semi-connected} \Rightarrow \text{connected.} \quad (5.1)$$

But, the converses need not be true as the following simple examples show.

**Example 5.7.** (1) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \tau)$ is connected but not semi-connected.

(2) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b, c\}\}$. Then $(X, \tau)$ is semi-connected but not $\beta$-connected.

**Lemma 5.8.** For a topological space $X$, the following properties are equivalent:

1. $X$ is $\beta$-connected or semipreconnected.
2. The intersection of two nonempty semipreopen subsets of $X$ is always nonempty.
3. The intersection of two nonempty preopen subsets of $X$ is always nonempty.
ON \( \partial \)-PRECONTINUOUS FUNCTIONS

(4) \( \text{pCl}(V) = X \) for every nonempty preopen subset \( V \) of \( X \).

(5) \( \text{spCl}(V) = X \) for every nonempty semipreopen subset \( V \) of \( X \).

**Proof.** The proofs of equivalences of (1), (2), and (3) are given in [2, Theorem 6.4].

The other properties (4) and (5), which are stated in [18], are easily equivalent to (3) and (2), respectively. \( \square \)

**Theorem 5.9.** If \( f : X \to Y \) is a \( \theta \)-p.c. surjection and \( X \) is \( \beta \)-connected, then \( Y \) is semi-connected.

**Proof.** Let \( V \) be any nonempty open set of \( Y \). Let \( y \in V \). Since \( f \) is surjective, there exists \( x \in X \) such that \( f(x) = y \). Since \( f \) is \( \theta \)-p.c., there exists \( U \in \text{PO}(X,x) \) such that \( f(\text{pCl}(U)) \subseteq \text{Cl}(V) \). Since \( X \) is \( \beta \)-connected, by Lemma 5.8 \( \text{pCl}(U) = X \) and hence \( \text{Cl}(V) = Y \) since \( f \) is surjective. Therefore, it follows from [22, Theorem 4.3] that \( Y \) is semi-connected. \( \square \)

**Remark 5.10.** The following example shows that the image of \( \beta \)-connectedness under weakly precontinuous surjections is not necessarily semi-connected.

**Example 5.11.** Let \( X \) be the set of real numbers, \( \tau = \{ \emptyset \} \cup \{ V \subseteq X : 0 \in V \} \), \( Y = \{ a, b, c \} \), and \( \sigma = \{ Y, \emptyset, \{ a \}, \{ b \}, \{ a, b \} \} \). Define a function \( f : (X,\tau) \to (Y,\sigma) \) as follows: \( f(x) = a \) if \( x < 0 \); \( f(x) = c \) if \( x = 0 \); \( f(x) = b \) if \( x > 0 \). Then \( f \) is a weakly precontinuous surjection which is not \( \theta \)-p.c. The topological space \( (X,\tau) \) is \( \beta \)-connected by Lemma 5.8. By Example 5.7(1), \( (Y,\sigma) \) is connected but not semi-connected. \( \square \)

**References**


TAKASHI NOIRI: DEPARTMENT OF MATHEMATICS, YATSUSHIRO COLLEGE OF TECHNOLOGY, YATSUSHIRO, KUMAMOTO, 866-8501, JAPAN
E-mail address: noiri@as.yatsushiuro-nct.ac.jp