PROPER CONTRACTIONS AND INVARIANT SUBSPACES

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ABSTRACT. Let $T$ be a contraction and $A$ the strong limit of \( \{T^nT^*\}_{n \geq 1} \). We prove the following theorem: if a hyponormal contraction $T$ does not have a nontrivial invariant subspace, then $T$ is either a proper contraction of class $\mathcal{C}_{0\varnothing}$ or a nonstrict proper contraction of class $\mathcal{C}_{10}$ for which $A$ is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*, T]$ is a strict contraction.

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1. Introduction. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. The null operator and the identity on $\mathcal{H}$ will be denoted by $O$ and $I$, respectively. If $T$ is an operator, then $T^*$ is its adjoint, and $\|T^*\| = \|T\|$. The null space (kernel) of $T$, which is the subspace of $\mathcal{H}$, will be denoted by $\mathcal{N}(T)$. A contraction is an operator $T$ such that $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for every $x$ in $\mathcal{H}$ or, equivalently, $T^*T \leq I$). A strict contraction is an operator $T$ such that $\|T\| < 1$ (i.e., $\sup_{\|x\|=1} \|Tx\|/\|x\| < 1$; equivalently, $T^*T < I$), which means that $T^*T \leq yI$ for some $y \in (0, 1))$. An isometry is a contraction for which $\|Tx\| = \|x\|$ for every $x$ in $\mathcal{H}$ (i.e., $T^*T = I$ so that $\|T\| = 1$).

We summarize below some well-known results on contractions that will be applied throughout (cf. [16, page 40], [5, 9, 10, 11, 13], and [8, Chapter 3]). If $T$ is a contraction, then $T^nT^* \overset{\sigma}{\rightarrow} A$. That is, the sequence $\{T^nT^*\}_{n \geq 1}$ of operators on $\mathcal{H}$ converges strongly to an operator $A$ on $\mathcal{H}$, which means that $\|(T^nT^* - A)x\| \to 0$ for every $x$ in $\mathcal{H}$. Moreover, $A$ is a nonnegative contraction (i.e., $O \leq A \leq I$), $\|A\| = 1$ whenever $A \neq O$, $T^nAT^n = A$ for every integer $n \geq 1$ (so that $T$ is an isometry if and only if $A = I$), $\|T^n\| = \|A^{1/2}\|$ for every $x$ in $\mathcal{H}$, and the null spaces of $A$ and $I - A$, viz. $\mathcal{N}(A) = \{x \in \mathcal{H}: Ax = 0\}$ and $\mathcal{N}(I - A) = \{x \in \mathcal{H}: Ax = x\}$, are given by

\[
\mathcal{N}(A) = \{x \in \mathcal{H}: T^n x \to 0\},
\]

\[
\mathcal{N}(I - A) = \{x \in \mathcal{H}: \|T^n x\| = \|x\| \quad \forall n \geq 1\}
\]

Recall that $T$ is a contraction if and only if $T^*$ is. Thus $T^nT^* \overset{\sigma}{\rightarrow} A_*$, where $O \leq A_* \leq I$, $\|A_*\| = 1$ whenever $A_* \neq O$, $T^nA_*T^n = A_*$ for every $n \geq 1$ (so that $T$ is a co-isometry—i.e., $T^*$ is an isometry—if and only if $A_* = I$), $\|T^n\| = \|A_*^{1/2}\|$ for every $x$ in $\mathcal{H}$, and

\[
\mathcal{N}(A_*) = \{x \in \mathcal{H}: T^n x \to 0\},
\]

\[
\mathcal{N}(I - A_*) = \{x \in \mathcal{H}: \|T^n x\| = \|x\| \quad \forall n \geq 1\}
\]
An operator $T$ on $\mathcal{H}$ is uniformly stable if the power sequence $\{T^n\}_{n=1}^{\infty}$ converges uniformly to the null operator (i.e., $\|T^n\| \to 0$). It is strongly stable if $\{T^n\}_{n=1}^{\infty}$ converges strongly to the null operator (i.e., $\|T^n x\| \to 0$ for every $x$ in $\mathcal{H}$), and weakly stable if $\{T^n\}_{n=1}^{\infty}$ converges weakly to the null operator (i.e., $\langle T^n x; y \rangle \to 0$ for every $x, y \in \mathcal{H}$ or, equivalently, $\langle T^n x; x \rangle \to 0$ for every $x \in \mathcal{H}$). It is clear that uniform stability implies strong stability, which implies weak stability. The converses fail (a unilateral shift is a weakly stable isometry and its adjoint is a strongly stable co-isometry) but hold for compact operators. $T$ is uniformly stable if and only if $T^*$ is uniformly stable, and $T$ is weakly stable if and only if $T^*$ is weakly stable. However, strong convergence is not preserved under the adjoint operation so that strong stability for $T$ does not imply strong stability for $T^*$ (and vice versa). If $T$ is a strongly stable contraction (i.e., if $\mathcal{N}(A) = \mathcal{H}$, which means that $A = O$), then it is usual to say that $T$ is a $\mathcal{C}_0$-contraction. If $T^*$ is a strongly stable contraction (i.e., if $\mathcal{N}(A_*) = \mathcal{H}$, which means that $A_*$ = $O$), then $T$ is a $\mathcal{C}_0$-contraction. On the other extreme, if a contraction $T$ is such that $T^n x \to 0$ for every nonzero vector $x$ in $\mathcal{H}$ (i.e., if $\mathcal{N}(A) = \{0\}$), then it is said to be a $\mathcal{C}_1$-contraction. Dually, if a contraction $T$ is such that $T^*_n x \to 0$ for every nonzero vector $x$ in $\mathcal{H}$ (i.e., if $\mathcal{N}(A_*) = \{0\}$), then it is a $\mathcal{C}_1$-contraction. These are the Nagy-Foiaş classes of contractions (see [16, page 72]). All combinations are possible leading to classes $\mathcal{C}_0$, $\mathcal{C}_0^1$, $\mathcal{C}_1$, and $\mathcal{C}_1^1$. In particular, $T$ and $T^*$ are both strongly stable contractions if and only if $T$ is of class $\mathcal{C}_0$. Generally,

\[
T \in \mathcal{C}_0 \iff A = A_* = O,
\]

\[
T \in \mathcal{C}_0^1 \iff A = O, \quad \mathcal{N}(A_*) = \{0\},
\]

\[
T \in \mathcal{C}_1 \iff \mathcal{N}(A) = \{0\}, \quad A_* = O,
\]

\[
T \in \mathcal{C}_1^1 \iff \mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}.
\]

If $T$ is a strict contraction, then it is uniformly stable, and hence of class $\mathcal{C}_0$. Thus, a contraction not in $\mathcal{C}_0$ is necessarily nonstrict (i.e., if $T \notin \mathcal{C}_0$, then $\|T\| = 1$). In particular, contractions in $\mathcal{C}_1$ or in $\mathcal{C}_1^1$ are nonstrict.

2. **Proper contractions.** An operator $T$ is a proper contraction if $\|Tx\| < \|x\|$ for every nonzero $x$ in $\mathcal{H}$ or, equivalently, if $T^* T < I$. The terms “strict” and “proper” contractions are sometimes interchanged in current literature. We adopt the terminology of [7, page 82] for strict contraction. Obviously, every strict contraction is a proper contraction, every proper contraction is a contraction, and the converses fail: any isometry is a contraction but not a proper contraction, and the diagonal operator $T = \text{diag} \{ (k+1)(k+2)^{-1} \}_{k=0}^{\infty}$ is a proper contraction on $\ell^2_2$ but not a strict contraction. Thus, proper contractions comprise a class of operators that is properly included in the class of all contractions and properly includes the class of all strict contractions. If $T$ is a proper contraction, then so is $T^* T$ (reason: $ST$ is a proper contraction whenever $S$ is a contraction and $T$ is a proper contraction). Thus, the point spectrum $\sigma_p(T^* T)$ lies in the open unit disc. If, in addition, $T$ is compact, then so is $T^* T$ and hence its spectrum $\sigma(T^* T)$, which is always closed, also lies in the open unit disc (for $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$ whenever $K$ is compact). This implies that the spectral radius $r(T^* T)$ is less than one. Therefore, $\|T\|^2 = r(T^* T) < 1$. 

**CONCLUSION.** The concepts of proper and strict contraction coincide for compact operators.

Proper contractions have been investigated in connection with unitary dilations (the minimal unitary dilation of a proper contraction is a bilateral shift whose multiplicity does not exceed the dimension of $\mathcal{H}$—see [16, page 91]), and also with strong stability of contractive semigroups (cf. [1]). They were further investigated in [15] by considering different topologies in $\mathcal{H}$. Here are three basic properties of proper contractions that will be needed in the sequel.

**PROPOSITION 2.1.** $T$ is a proper contraction if and only if $T^*$ is a proper contraction.

**Proof.** Recall that $\|T^*x\|^2 = \langle T^*x; T^*x \rangle = \langle TT^*x; x \rangle \leq \|TT^*x\| \|x\|$ for every $x$ in $\mathcal{H}$, for all operators $T$ on $\mathcal{H}$. Take an arbitrary nonzero vector $x$ in $\mathcal{H}$. If $T^*x = 0$, then $\|T^*x\| < \|x\|$ trivially. On the other hand, if $T^*x \neq 0$ and $T$ is a proper contraction, then $\|TT^*x\| < \|T^*x\| \neq 0$ so that $\|T^*x\|^2 < \|T^*x\| \|x\|$, and hence $\|T^*x\| < \|x\|$. That is, $T^*$ is a proper contraction. Dually, since $T^{**} = T$, it follows that $T$ is a proper contraction whenever $T^*$ is.

If $S$ is a contraction and $T$ is a proper contraction, then $ST$ is a proper contraction (as we have already seen above) and so is $S^*T^*$ by Proposition 2.1. Another application of Proposition 2.1 ensures that $TS = (S^*T^*)^*$ is still a proper contraction. Summing up: left or right product of a contraction and a proper contraction is again a proper contraction.

**PROPOSITION 2.2.** Every proper contraction is weakly stable.

**Proof.** If $\|Tx\| < \|x\|$ for every nonzero $x$ in $\mathcal{H}$, then $T$ is completely nonisometric (i.e., there is no nonzero reducing subspace $\mathcal{M}$ for $T$ such that $\|T^n x\| = \|x\|$ for every $x \in \mathcal{M}$ and every $n \geq 1$), and therefore completely nonunitary. But a completely nonunitary contraction is weakly stable. In fact, the Foguel decomposition for contractions says that every contraction is the direct sum of a weakly stable contraction and a unitary operator (cf. [6, page 55] or [8, page 106]).

The converse of Proposition 2.2 fails: shifts are weakly stable isometries. However, as it was raised in [1], a proper contraction is not necessarily strongly stable. Indeed, if $T$ is the weighted unilateral shift $T = \text{shift}_{\infty}(k + 1)^{1/2}(k + 2)^{-1}(k + 3)^{1/2})_{k=0}^\infty$ on $\ell^2_+$, which is a proper contraction because $(k + 1)(k + 2)^{-1}(k + 3) < 1$ for every $k \geq 0$, then $A$ is the diagonal operator $A = \text{diag}_{\infty}(k + 1)(k + 2)^{-1})_{k=0}^\infty \neq O$ (cf. [10] or [8, pages 51, 52]) so that $T$ is not strongly stable. As a matter of fact, $\mathcal{N}(A) = \{0\}$ and (as it is readily verified) $A^* = O$. Hence $T$ is a proper contraction of class $\mathcal{C}_{10}$. The converse is much simpler: strongly stable contractions are not necessarily proper contractions. For instance, a backward unilateral shift $S^+_n$ is a strongly stable co-isometry (in fact, an operator is a strongly stable co-isometry if and only if it is a backward unilateral shift). Thus $S^+_n$ is a strongly stable contraction but not a proper contraction (it is a nonproper contraction of class $\mathcal{C}_{01}$). Actually, even a $\mathcal{C}_{00}$-contraction is not necessarily a proper contraction. For example, the weighted bilateral shift $T = \text{shift}_{\infty}(k + 1)^{1/2}(k + 2)^{-1}_k=\infty$ on $\ell^2$ is a contraction of class $\mathcal{C}_{00}$ (reason: $\prod_k^n(|k| + 1)^{-1} = (n!)^{-1} \to 0$ as $n \to \infty$, which means that both products $\prod_{k=0}^n(|k| + 1)^{-1}$ and $\prod_{k=\infty}^n(|k| + 1)^{-1}$ diverge to...
0—see [3, page 181]) but not a proper contraction because \((|k| + 1)^{-1} = 1\) for \(k = 0\). It is worth noticing that the weighted bilateral shift \(T = \text{shift}\{1 - ((|k| + 2)^{-2})_{k=-\infty}^\infty\}\) on \(\ell^2\) is a proper contraction of class \(\ell_{11}\). Indeed, \(0 < 1 - ((|k| + 2)^{-2} < 1\) for each integer \(k\), and both products \(\prod_{k=0}^\infty (1 - ((|k| + 2)^{-2})\) and \(\prod_{k=-\infty}^0 (1 - ((|k| + 2)^{-2})\) do not diverge to 0 (cf. [3, page 181] again)—these products converge once the series \(\sum_{k=0}^\infty ((|k| + 2)^{-2}\) converges.

**PROPOSITION 2.3.** If \(T\) is a proper contraction, then \(A\) is a proper contraction.

**Proof.** Let \(T\) be a proper contraction and take an arbitrary nonzero vector \(x\) in \(\mathbb{H}\). If \(T^m x = 0\) for some \(m \geq 1\), then \(T^n x = 0\) for every integer \(n \geq m\). If \(T^m x \neq 0\) for every integer \(n \geq 1\), then \(\|T^{n+1} x\| = \|TT^n x\| < \|T^n x\| < \|x\|\) so that \(\{\|T^n x\|\}_{n=1}^\infty\) is a strictly decreasing sequence of positive numbers. In the former case \(T\) is trivially strongly stable so that \(A = O\) is a trivial proper contraction. In the latter case \(\{\|T^n x\|\}_{n=1}^\infty\) converges in the real line to \(\|A^{1/2} x\|\) so that \(\|A^{1/2} x\| < \|x\|\). Thus \(\|Ax\| \leq \|A^{1/2} x\| < \|x\|\). □

A backward unilateral shift shows that the converse of **Proposition 2.3** does not hold true as well (i.e., there exist nonproper contractions \(T\) for which \(A\) is a proper contraction).

3. Invariant subspaces. A subspace \(\mathcal{M}\) of \(\mathbb{H}\) is a closed linear manifold of \(\mathbb{H}\). \(\mathcal{M}\) is nontrivial if \(\{0\} \neq \mathcal{M} \neq \mathbb{H}\). If \(T\) is an operator on \(\mathbb{H}\) and \(T(\mathcal{M}) \subseteq \mathcal{M}\), then \(\mathcal{M}\) is invariant for \(T\) (or \(\mathcal{M}\) is \(T\)-invariant). If \(\mathcal{M}\) is a nontrivial invariant subspace for \(T\), then its orthogonal complement \(\mathcal{M}^\perp\) is a nontrivial invariant subspace for \(T^*\). If \(\mathcal{M}\) is invariant for both \(T\) and \(T^*\) (equivalently, if both \(\mathcal{M}\) and \(\mathcal{M}^\perp\) are \(T\)-invariant), then \(\mathcal{M}\) reduces \(T\). A classical open question in operator theory is: does a contraction not in \(\ell_{00}\) have a nontrivial invariant subspace? Although this is still an unsolved problem we know that the following result holds true.

**Lemma 3.1.** If a contraction has no nontrivial invariant subspace, then it is either a \(\ell_{00}\), a \(\ell_{01}\), or a \(\ell_{10}\)-contraction.

**Proof.** See, for instance, [8, page 71]. □

The class of contractions \(T\) for which \(A\) is a projection was investigated in [4, 10]. It coincides with the class of all contractions \(T\) that commute with \(A\); that is, \(A = A^2\) if and only if \(AT = TA\) (cf. [4]). Equivalently, \(\mathcal{N}(A - A^2) = \mathbb{H}\) if and only if \(\mathcal{N}(AT - TA) = \mathbb{H}\). The next proposition extends this equivalence.

**Proposition 3.2.** \(\mathcal{N}(A - A^2)\) is the largest subspace of \(\mathbb{H}\) that is included in \(\mathcal{N}(AT - TA)\) and is \(T\)-invariant.

**Proof.** See [10] (or [8, page 52]). □

We will say that \(A\) is completely nonprojective if \(Ax \neq A^2 x\) for every nonzero \(x\) in \(\mathbb{H}\) (i.e., if \(\mathcal{N}(A - A^2) = \{0\}\)). Since \(\mathcal{N}(A - A^2)\) reduces the selfadjoint operator \(A\), this means that no nonzero direct summand of \(A\) is a projection. If \(A\) is completely nonprojective, then \(T\) is a \(\ell_{1.1}\)-contraction (for \(\mathcal{N}(A) \subseteq \mathcal{N}(A - A^2)\)).
**Lemma 3.3.** If a contraction $T$ has no nontrivial invariant subspace, then either $T$ is strongly stable or $A$ is a completely nonprojective nonstrict proper contraction.

**Proof.** Suppose that $T$ is a contraction without a nontrivial invariant subspace. Since $\mathcal{N}(A - A^2)$ is an invariant subspace for $T$ (by Proposition 3.2), it follows that either $\mathcal{N}(A - A^2) = \mathcal{H}$ or $\mathcal{N}(A - A^2) = \{0\}$. In the former case $A$ is a projection (i.e., $A = A^2$). However, as it was shown in [10], if $A$ is a projection then $T$ is the direct sum of a strongly stable contraction $G$, a unilateral shift $S_+$, and a unitary operator $U$, where any of the direct summands of the decomposition

$$T = G \oplus S_+ \oplus U$$

may be missing (see also [8, page 83]). But $T$ has no nontrivial invariant subspace so that $T = G$. That is, $T$ is a strongly stable contraction, for $S_+$ and $U$ clearly have nontrivial invariant subspaces (isometries have nontrivial invariant subspaces). In the latter case $A$ is a completely nonprojective proper contraction. Indeed, $\{x \in \mathcal{H} : ||Ax|| = ||x||\} = \mathcal{N}(I - A) \subseteq \mathcal{N}(A - A^2) = \{0\}$. Finally, the contraction $A$ is not strict (i.e., $||A|| = 1$) whenever $T$ is not strongly stable (i.e., whenever $A \neq O$).

Another classical open question in operator theory is: does a hyponormal operator have a nontrivial invariant subspace? Recall that an operator $T$ on $\mathcal{H}$ is hyponormal if $TT^* \leq T^*T$ (equivalently, if $||T^*x|| \leq ||Tx||$ for every $x$ in $\mathcal{H}$), and $T$ is cohyponormal if $T^*$ is hyponormal. Here is a consequence of Lemmas 3.1 and 3.3 for hyponormal contractions. It uses the fact that a cohyponormal contraction $T$ is such that $A$ is a projection. This implies that a completely nonunitary cohyponormal contraction is strongly stable (cf. [9, 12, 14]).

**Theorem 3.4.** If a hyponormal contraction $T$ has no nontrivial invariant subspace, then it is either a $\epsilon_00$-contraction or a $\epsilon_10$-contraction for which $A$ is a completely nonprojective nonstrict proper contraction.

**Proof.** If $T$ has no nontrivial invariant subspace, then $T^*$ has no nontrivial invariant subspace. If $T$ is a contraction, then Lemmas 3.1 and 3.3 ensure that either $A = A_* = O$, $A = O$ and $A_*$ is a completely nonprojective nonstrict proper contraction, or $A$ is a completely nonprojective nonstrict proper contraction and $A_* = O$. However, if $T$ is hyponormal, then $A_*$ is a projection [9] so that $A_* = O$ (see also [8, page 78]).

Can the conclusion in Theorem 3.4 be sharpened to $T \in \epsilon_00$? In other words, does a hyponormal contraction not in $\epsilon_00$ have a nontrivial invariant subspace? The question has an affirmative answer if we replace “$\epsilon_00$-contraction” with “proper contraction.” That is, if a hyponormal contraction is not a proper contraction, then it has a nontrivial invariant subspace. This will be proved in Theorem 3.6 below, but first we consider the following auxiliary result. Let $D$ denote the self-commutator of $T$; that is,


Thus, a hyponormal is precisely an operator $T$ for which $D$ is nonnegative (i.e., $D \geq 0$).

**Proposition 3.5.** If $T$ is a hyponormal contraction, then $D$ is a contraction whose power sequence converges strongly. If $P$ is the strong limit of $\{D^n\}_{n \geq 1}$, then $PT = O$. 

\[ D = [T^*, T] = T^*T - TT^*. \]  

(3.2)
**Proof.** Take an arbitrary \( x \) in \( \mathcal{H} \) and an arbitrary nonnegative integer \( n \). Suppose that \( T \) is hyponormal and let \( R = D^{1/2} \geq O \) be the unique nonnegative square root of \( D \geq O \). If, in addition, \( T \) is a contraction, then

\[
\langle D^{n+1} x; x \rangle = \| R^{n+1} x \|^2 = \langle DR^n x; R^n x \rangle \\
= \| TR^n x \|^2 - \| T^* R^n x \|^2 \\
\leq \| R^n x \|^2 - \| T^* R^n x \|^2 \leq \| R^n x \|^2 \\
= \langle D^n x; x \rangle.
\]

This shows that \( R \) (and so \( D \)) is a contraction: set \( n = 0 \) above. It also shows that \( \{D^n\}_{n \geq 1} \) is a decreasing sequence of nonnegative contractions. Since a bounded monotone sequence of selfadjoint operators converges strongly,

\[
D^n \xrightarrow{s} P \geq O.
\]

Indeed, the strong limit \( P \) of \( \{D^n\}_{n \geq 1} \) is nonnegative, for the set of all nonnegative operators on \( \mathcal{H} \) is weakly (thus strongly) closed. As a matter of fact, \( P = P^2 \) (the weak limit of any weakly convergent power sequence is idempotent) and so \( P \geq O \) is a projection. Moreover,

\[
\sum_{n=0}^{m} \| T^* R^n x \|^2 \leq \sum_{n=0}^{m} (\| R^n x \|^2 - \| R^{n+1} x \|^2) = \| x \|^2 - \| R^{m+1} x \|^2 \leq \| x \|^2
\]

for all \( m \geq 0 \) so that \( \| T^* R^n x \| \to 0 \) as \( n \to \infty \). Hence

\[
T^* P x = T^* \lim_n D^n x = \lim_n T^* R^{2n} x = 0
\]

for every \( x \) in \( \mathcal{H} \), and therefore \( PT = O \) (since \( P \) is selfadjoint).

**Theorem 3.6.** If a hyponormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction.

**Proof.** (a) Take an arbitrary operator \( T \) on \( \mathcal{H} \) and an arbitrary \( x \) in \( \mathcal{H} \). Note that

\[
T^* T x = \| T \|^2 x \text{ if and only if } \| T x \| = \| T \| \| x \|.
\]

Indeed, if \( T^* T x = \| T \|^2 x \), then \( \| T x \|^2 = \langle T^* T x; x \rangle = \| T \|^2 \| x \|^2 \). Conversely, if \( \| T x \| = \| T \| \| x \| \), then \( \langle T^* T x; T^* T x \rangle = \| T \|^4 \| x \|^2 \), and hence

\[
\| T^* T x \|^2 = \| T \|^4 \| x \|^2 - 2 \text{ Re} \langle T^* T x; T^* T x \rangle + \| T \|^4 \| x \|^2 \\
= \| T^* T x \|^2 - \| T \|^4 \| x \|^2 \leq (\| T \|^4 - \| T \|^4) \| x \|^2 = 0.
\]

Put \( \mathcal{M} = \{ x \in \mathcal{H} : \| T x \| = \| T \| \| x \| \} = \mathcal{N}(\| T \|^2 I - T^* T) \), which is a subspace of \( \mathcal{H} \). If \( T \) is hyponormal, then \( \mathcal{M} \) is \( T \)-invariant. In fact, if \( T \) is hyponormal and \( x \in \mathcal{M} \), then

\[
\| T( T x) \| \leq \| T \| \| T x \| = \| T \| \| x \| = \| T^* T x \| \leq \| T( T x) \|
\]

(3.9)
and so $Tx \in \mathcal{M}$ (see also [6, page 9]). Now let $T$ be a hyponormal contraction. If $\|T\| < 1$, then it is trivially a proper contraction. If $\|T\| = 1$ and $T$ has no nontrivial invariant subspace, then $\mathcal{M} = \{x \in \mathcal{H} : \|Tx\| = \|x\|\} = \{0\}$ (actually, if $\mathcal{M} = \mathcal{H}$, then $T$ is an isometry, and isometries have invariant subspaces). Hence $T$ is a proper contraction.

(b) Let $D \geq O$ be the self-commutator of a hyponormal contraction $T$ and let $P$ be the strong limit of $\{D^n\}_{n \geq 1}$ so that $PT = O$ (cf. Proposition 3.5). Suppose $T$ has no nontrivial invariant subspace. Since $\mathcal{N}(P)$ is a nonzero invariant subspace for $T$ whenever $PT = O$ and $T \neq O$, it follows that $\mathcal{N}(P) = \mathcal{H}$. Hence $P = O$ and so $D$ is strongly stable ($D^n \rightarrow O$). Moreover, since $\bigvee \{T^n x\}_{n \geq 0}$ is a nonzero invariant subspace for $T$ whenever $x \neq 0$, it follows that $\bigvee \{T^n x\}_{n \geq 0} = \mathcal{H}$ for each $x \neq 0$ (every nonzero vector in $\mathcal{H}$ is a cyclic vector for $T$). Thus the Berger-Shaw theorem (see, for instance, [2, page 152]) ensures that $D$ is a trace-class operator so that $D$ is compact (i.e., $T$ is essentially normal). But for compact operators strong stability coincides with uniform stability, and uniform stability always means spectral radius less than one. Hence the nonnegative $D$ is a strict contraction because it is clearly normaloid (i.e., $\|D\| = r(D) < 1$).

**Remark 3.7.** According to the Berger-Shaw theorem, a hyponormal contraction without a nontrivial invariant subspace has a trace-class self-commutator $D$ with trace-norm $\|D\|_1 \leq 1$. If $D \neq O$ is not a rank-one operator, then $\|D\| < \|D\|_1 \leq 1$. The above argument ensures the inequality $\|D\| < 1$ whenever a hyponormal contraction has no nontrivial invariant subspace, including the case of a hyponormal contraction with a rank-one self-commutator.

An operator is seminormal if it is hyponormal or cohyponormal. Recall that $T^*$ has a nontrivial invariant subspace if and only if $T$ has, $T^*$ is a proper contraction if and only if $T$ is (Proposition 2.1), and $[T, T^*] = -[T^*, T]$. Thus, the above theorem also holds for cohyponormal contractions. If a seminormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction. This prompts the question: can we drop “hyponormal” from the theorem statement? In particular, is it true that every nonproper contraction has a nontrivial invariant subspace? Theorems 3.4 and 3.6 yield the following result.

**Corollary 3.8.** If a hyponormal contraction $T$ has no nontrivial invariant subspace, then it is either a proper contraction of class $\mathcal{C}_{00}$ or a nonstrict proper contraction of class $\mathcal{C}_{10}$ for which $A$ is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*, T]$ is a strict contraction.

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