ON THE STRONGLY STARLIKENESS OF MULTIVALENTLY CONVEX FUNCTIONS OF ORDER \( \alpha \)

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Abstract. The object of the present paper is to derive some sufficient conditions for strongly starlikeness of multivalently convex functions of order \( \alpha \) in the open unit disc.

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1. Introduction. Let \( \mathcal{A}(p) \) denote the class of the functions \( f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \) which are analytic in the open unit disc \( \mathbb{E} = \{ z : |z| < 1 \} \). A function \( f(z) \in \mathcal{A}(p) \) is called \( p \)-valently starlike if and only if the inequality

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0
\]

holds for \( z \in \mathbb{E} \). A function \( f(z) \in \mathcal{A}(p) \) is called \( p \)-valently convex of order \( \alpha \) \((0 \leq \alpha < p)\) if and only if the inequality

\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha
\]

holds for \( z \in \mathbb{E} \). We denote by \( \mathcal{A}(p, \alpha) \) the family of such functions. A function \( f(z) \in \mathcal{A}(p) \) is said to be strongly starlike of order \( \alpha \) \((0 < \alpha \leq 1)\) if and only if the inequality

\[
\left| \text{arg} \left\{ \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi}{2} \alpha
\]

holds for \( z \in \mathbb{E} \). We also denote by \( \text{STS}(p, \alpha) \) the family of functions which satisfy the above inequality for the argument. From the definition, it follows that if \( f(z) \in \text{STS}(p, \alpha) \), then we have

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0\quad \text{in } \mathbb{E}
\]

or \( f(z) \) is \( p \)-valently starlike in \( \mathbb{E} \) and therefore \( f(z) \) is \( p \)-valent in \( \mathbb{E} \) (see [1, Lemma 7]). Nunokawa [2, 3] proved the following theorems.

**Theorem 1.1** (see [2]). If \( f(z) \in \mathcal{A}(p) \) satisfies

\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < p + \frac{\alpha}{2},
\]

where \( 0 < \alpha \leq 1 \), then \( f(z) \in \text{STS}(p, \alpha) \).
Theorem 1.2 (see [3]). If \( f(z) \in \mathcal{A}(1) \) satisfies
\[
\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in } \mathcal{E},
\]
(1.6)
then
\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \beta \quad \text{in } \mathcal{E},
\]
(1.7)
where
\[
\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\beta q(\beta) \sin(\pi/2)(1-\beta)}{p(\beta) + \beta q(\beta) \cos(\pi/2)(1-\beta)} \right\},
\]
(1.8)
\[
p(\beta) = (1+\beta)^{(1+\beta)/2}, \quad q(\beta) = (1-\beta)^{(\beta-1)/2}.
\]

It is the purpose of the present paper to prove that if \( f(z) \in \mathcal{C}(1,1-(\alpha/2)) \), then \( f(z) \in \text{STS}(1,\alpha) \).

In this paper, we need the following lemma.

Lemma 1.3. Let \( f(z) \in \mathcal{A}(1) \) be starlike with respect to the origin in \( \mathcal{E} \). Let \( C(r,\theta) = \{f(te^{i\theta}): 0 \leq t \leq r < 1\} \) and \( T(r,\theta) \) be the total variation of \( \arg f(te^{i\theta}) \) on \( C(r,\theta) \), so that
\[
T(r,\theta) = \int_0^r \left| \frac{d}{dt} \arg \{f(te^{i\theta})\} \right| dt.
\]
(1.9)
Then
\[
T(r,\theta) < \pi.
\]
(1.10)
We owe this lemma to Sheil-Small [6, Theorem 1].

2. Main theorem. Our main theorem for the starlikeness of multivalently convex functions of order \( \alpha \) is the following.

Theorem 2.1. Let \( f(z) \in \mathcal{A}(1) \) and
\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 1 - \frac{\alpha}{2} \quad \text{in } \mathcal{E},
\]
(2.1)
where \( 0 < \alpha \leq 1 \). Then
\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathcal{E},
\]
(2.2)
or \( f(z) \) is strongly starlike of order \( \alpha \) in \( \mathcal{E} \).

Proof. We put
\[
\frac{2}{\alpha} \left\{ 1 + \frac{zf''(z)}{f'(z)} - 1 + \frac{\alpha}{2} \right\} = \frac{zg'(z)}{g(z)},
\]
(2.3)
where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. From assumption (2.1), we have
\[
\Re \left\{ \frac{zg'(z)}{g(z)} \right\} > 0 \quad \text{in } \mathbb{C}.
\] (2.4)

This shows that $g(z)$ is starlike and univalent in $\mathbb{C}$. With an easy calculation (cf. [4]), (2.3) gives us that
\[
f'(z) = \left\{ \frac{g(z)}{z} \right\}^{\alpha/2}.
\] (2.5)

Since
\[
f'(z) \neq 0, \quad 0 < |z| < 1,
\] (2.6)
we easily have
\[
\frac{f(z)}{zf'(z)} = \int_{0}^{1} \frac{f'(tz)}{f'(z)} \, dt = \int_{0}^{1} t^{-\alpha/2} \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}^{\alpha/2} \, dt,
\] (2.7)
where $z = re^{i\theta}$ and $0 < r < 1$. Since $g(z)$ is starlike in $\mathbb{C}$, from Lemma 1.3, we have
\[
-\pi < \arg \left\{ g(tre^{i\theta}) \right\} - \arg \left\{ g(re^{i\theta}) \right\} < \pi
\] (2.8)
for $0 < t \leq 1$. Putting
\[
\xi = \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}^{\alpha/2},
\] (2.9)
we have
\[
\arg s = \frac{\alpha}{2} \arg \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}.
\] (2.10)

From (2.8) and (2.10), $s$ lies in the convex sector
\[
\left\{ s : |\arg s| \leq \frac{\pi}{2} \right\}
\] (2.11)
and the same is true of its integral mean of (2.7), (cf. [5, Lemma 1]). Therefore, we have
\[
\left| \arg \left\{ \frac{f(z)}{zf'(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathbb{C}
\] (2.12)
or
\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathbb{C}.
\] (2.13)

This shows that
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathbb{C},
\] (2.14)
which completes the proof of our main theorem. □
**Remark 2.2.** This result is sharp for the case \( \alpha \to 0 \) and \( \alpha = 1 \).

(a) For the case \( \alpha \to 0 \), put \( f(z) = z \), then \( f(z) \) is a convex function of order \( 1 - (\alpha/2) \to 1 \) and \( f(z) \) then \( f(z) \) is a strongly starlike function of order \( \alpha \to 0 \).

(b) For the case \( \alpha = 1 \), put

\[
1 + \frac{z f''(z)}{f'(z)} = \frac{1}{1-z}.
\]

Then we have

\[
1 + \text{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} > \frac{1}{2} \quad \text{in} \, \mathbb{C},
\]

and therefore \( f(z) \) is a convex function of order \( 1/2 \). From (2.10), we easily have

\[
f''(z) = \frac{1}{1-z}, \quad f(z) = \log \left\{ \frac{1}{1-z} \right\}.
\]

Putting \( |z| = 1 \), \( z = e^{i\theta}, 0 \leq \theta < 2\pi \), then it follows that

\[
\frac{z}{1-z} = -\frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)},
\]

\[
\log \left\{ \frac{1}{1-z} \right\} = \log \left\{ \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\} + i \arg \left\{ \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\}.
\]

\[
\lim_{\theta \to +0} \arg \left\{ \frac{z f''(z)}{f'(z)} \right\} = \lim_{\theta \to +0} \arg \left\{ \frac{z/(1-z)}{\log(1/(1-z))} \right\}
\]

\[
= \lim_{\theta \to +0} \left\{ -\frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\}
\]

\[
- \lim_{\theta \to +0} \left\{ \log \left\{ \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\} + i \arg \left\{ \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\} \right\}
\]

\[
= \frac{\pi}{2}.
\]

The above shows that the main theorem is sharp for the case \( \alpha \to 0 \) and \( \alpha = 1 \).

Applying the same method as above and [2], we can obtain the following result.

**Theorem 2.3.** If \( f(z) \in A(p) \) and satisfies

\[
p - \frac{\alpha}{2} < 1 + \text{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} \quad \text{in} \, \mathbb{C},
\]

where \( 0 < \alpha \leq 1 \), then \( f(z) \in \text{STS}(p,\alpha) \).

**References**


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