EVALUATION OF EULER-ZAGIER SUMS

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Abstract. We present a simple method for evaluation of multiple Euler sums in terms of single and double zeta values.

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1. Introduction. We give a short evaluation of the triple sums

\[ w(p,q,r) = \sum_{n,m=1}^{\infty} \frac{1}{nm(n+m)^r} \]  \hspace{1cm} (1.1)

in terms of single zeta values \( \zeta(p) \)

\[ \zeta(p) = \sum_{n=1}^{\infty} n^{-p} \] \hspace{1cm} (1.2)

and double zeta values (Euler sums) \( S(p,q) \)

\[ S(p,q) = \sum_{n=1}^{\infty} H_n^{(p)} n^{-q}, \quad H_n^{(p)} = 1^{-p} + 2^{-p} + \cdots + n^{-p}, \] \hspace{1cm} (1.3)

where \( p \geq 1, q > 1 \).

Multiple Euler sums have been discussed and evaluated in a number of papers of which we want to point out \([1, 2, 3, 4, 5, 6, 7, 10]\). Also \([8, \text{Sections 18 and 19}]\). We refer to these publications for general comments and details.

2. Euler sums

\textbf{Lemma 2.1.} For any integer \( p > 1 \) and any \( x > 0 \),

\[ \sigma(p;x) = \sum_{n=1}^{\infty} \frac{1}{n^p(n+x)} = \sum_{k=1}^{p-1} \frac{(-1)^{k-1} \zeta(p-k+1)}{x^k} + \frac{(-1)^{p-1}}{x^p} (\psi(x+1) + \gamma), \] \hspace{1cm} (2.1)

where \( \psi = \Gamma'/\Gamma \) is the psi function and \( \gamma \) is Euler’s constant.

\textbf{Proof.} We have

\[ \sigma(p;x) = \sum_{n=1}^{\infty} \frac{x+n-n}{n^p x(n+x)} = \sum_{n=1}^{\infty} \frac{1}{n^p x} - \sum_{n=1}^{\infty} \frac{1}{n^{p-1} x(n+x)} \] \hspace{1cm} (2.2)

\[ = \frac{1}{x} (\zeta(p) - \sigma(p-1;x)). \]
Therefore, (see [9, page 665])

\[
\sigma(1; x) = \sum_{n=1}^{\infty} \frac{1}{n(n+x)} = \frac{1}{x}(\psi(x+1) + y).
\] (2.3)

Now we differentiate (2.1) \( r-1 \) times, where \( r > 1 \). With \( D = d/dx \) we have

\[
D^{r-1} \frac{1}{n+x} = \frac{(-1)^{r-1}(r-1)!}{(n+x)^r},
\]

\[
D^{r-1} \frac{1}{x^k} = (-1)^{r-1} \frac{(r+k-2)!}{(k-1)!} \frac{1}{x^{r+k-1}} = (-1)^{r-1}(r-1) \left( \frac{r+k-2}{r-1} \right) \frac{1}{x^{r+k-1}},
\] (2.4)

\[
D^{r-1}(x^{-p}(\psi(x+1) + y)) = \sum_{k=0}^{r-1} \binom{r-1}{k} (D^{r-1-k}x^{-p})(D^k(\psi(x+1) + y)).
\]

Therefore,

\[
D^{r-1}\sigma(p; x) = (-1)^{r-1}(r-1)! \sum_{n=1}^{\infty} \frac{1}{n^p(n+x)^r}
\]

\[
= (-1)^{r-1}(r-1)! \sum_{k=1}^{p-1} (-1)^{k-1} \binom{r+k-2}{r-1} \frac{\zeta(p-k+1)}{x^{k+r-1}}
\]

\[
+ \frac{(-1)^{p-1}(p-1)!}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!} \frac{(\psi(x+1)+y)^{(k)}}{x^{r+p-k-1}}.
\] (2.5)

We summarize this result in the following lemma.

**Lemma 2.2.** For any integers \( p > 1, r \geq 1 \) and any \( x > 0 \),

\[
\sum_{n=1}^{\infty} \frac{1}{n^p(n+x)^r} = \sum_{k=1}^{p-1} (-1)^{k-1} \binom{r+k-2}{r-1} \frac{\zeta(p-k+1)}{x^{k+r-1}}
\]

\[
+ \frac{(-1)^{p-1}(p-1)!}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!} \frac{(\psi(x+1)+y)^{(k)}}{x^{r+p-k-1}}.
\] (2.6)

Next, we replace here \( x \) by \( mx \) and multiply both sides by \( m^{-q} \), \( q \geq 1 \). This gives

\[
\sum_{n=1}^{\infty} \frac{1}{n^p m^q(n+mx)^r} = \sum_{k=0}^{p-2} (-1)^k \binom{r+k-1}{r-1} \frac{\zeta(p-k)}{m^{k+q+r}}
\]

\[
+ \frac{(-1)^{p-1}(p-1)!}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!} \frac{1}{m^{r+p-k-1}} \frac{(\psi(mx+1)+y)^{(k)}}{m^{r+p-q-k-1}}.
\] (2.7)

Summing for \( m = 1, 2, \ldots \) we obtain our main representation.
Theorem 2.3. For all integers \( p > 1, r \geq 1 \) and all \( q \geq 0, q + r > 1, x > 0 \),

\[
\sigma(p,q,r;x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n + m x)^r} = \sum_{k=0}^{p-2} \frac{(-1)^k}{x^{k+r}} \binom{r+k-1}{r-1} \zeta(p-k) \zeta(k+q+r) + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k (r+p-k-2)!}{k!(r-k-1)!} \frac{1}{x^{r+p-k-1}} \sum_{m=1}^{\infty} \frac{(\psi(mx+1)+y)^{(k)}}{m^{r+p-q-k-1}}.
\]

The case \( p = 1 \) can be derived directly from (2.3), namely,

\[
\sigma(1,q,r;x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n m^q (n + m x)^r} = \sum_{k=0}^{r-1} \frac{(-1)^k}{k!} \frac{1}{x^{r-k}} \sum_{m=1}^{\infty} \frac{(\psi(mx+1)+y)^{(k)}}{m^{r+q-k}}.
\]

We remind the reader that the expression \((\psi(mx+1)+y)^{(k)}\) stands for the \(k\)th derivative of the function \(\psi(x+1)+y\) evaluated at \(mx\).

By setting \(x = 1\) we get the desired representation of \(w(p,q,r)\). Making use of

\[
\psi(m+1)+y = H_{m}^{(1)} = 1 + 2^{-1} + \cdots + m^{-1},
\]

\[
\psi^{(k)}(m+1) = (-1)^k k! [H_{m}^{(k+1)} - \zeta(k+1)],
\]

(see [9, page 775]), and with the agreement to read \(\zeta(1) = 0\), one obtains the following corollary.

Corollary 2.4. For all integers \( p > 1, r \geq 1 \) and all \( q \geq 0 \) with \( q + r > 1 \),

\[
w(p,q,r) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n + m)^r} = \sum_{k=0}^{p-2} \frac{(-1)^k}{x^{k+r}} \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+q+k) + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(r+p-k-2)!}{(r-k-1)!} [S(k+1,r+p+q-k-1) - \zeta(k+1) \zeta(r+p+q-k-1)],
\]

(2.11)

in particular,

\[
w(1,q,r) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n m^q (n + m)^r} = \sum_{k=0}^{r-1} [S(k+1,r+q-k) - \zeta(k+1) \zeta(r+q-k)].
\]

(2.12)
When \( q > 0 \) (or \( q \geq 1, \ p = 1 \)) we also have

\[
\begin{align*}
w(p, q, 1) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n + m)} \\
&= \frac{p-1}{(p-1)!} \sum_{k=1}^{p-1} (-1)^{k-1} \zeta(p-k+1) \zeta(q+k) + (-1)^{p-1} S(1, p + q).
\end{align*}
\]

(2.13)

3. Remarks. Our notation \( S(p, q) \) corresponds to \( S_{p,q} \) in [5]. The authors of [2] use the sums \( \zeta(p, q) \), which equal \( S(q, p) \), which equal \( S(q, p) - \zeta(p + q) \).

The representation (2.11) has strong and weak points. One good feature is that \( q \) need not be an integer. A weak point is that the right-hand side in (2.11) is not explicitly symmetrical in \( p \) and \( q \), while obviously \( w(p, q, r) = w(q, p, r) \). Moreover, the right-hand side has too many terms. For instance, when \( q = 0 \) (2.11) becomes

\[
\begin{align*}
w(p, 0, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p (n + m)^r} \\
&= (-1)^{p-1} S(r, p) + \sum_{k=0}^{p-2} (-1)^k \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+k) \\
& \quad + \frac{(-1)^{p-1}}{(p-1)! r-2} \sum_{k=0}^{r-2} \frac{(r+p-k-2)!}{(r-k-1)!} \{S(k+1, r+p-k-1) - \zeta(k+1) \zeta(r+p-k-1)\}
\end{align*}
\]

(3.1)

(here the term \((-1)^{p-1} S(r, p)\) is written separately on purpose).

At the same time

\[
w(p, 0, r) = \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{m=1}^{\infty} \frac{1}{(n + m)^r} = \sum_{n=1}^{\infty} \frac{1}{n^p} (\zeta(r) - H_{n}^{(r)}) = \zeta(p) \zeta(r) - S(r, p)
\]

(3.2)

which is much shorter. However, we can benefit from this situation if we compare the two representations of \( w(p, 0, r) \) and derive relations for the single and double Euler sums. For instance, when \( p \) is odd, we can solve for \( S(r, p) \) to get

\[
\begin{align*}
2S(r, p) &= \sum_{k=1}^{p-2} (-1)^{k+1} \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+k) \\
& \quad + \frac{(-1)^{p-1} r-2}{(p-1)!} \sum_{k=0}^{r-2} \frac{(r+p-k-2)!}{(r-k-1)!} \{S(k+1, r+p-k-1) - \zeta(k+1) \zeta(r+p-k-1)\}
\end{align*}
\]

(3.3)

that is, \( S(r, p) \) can be expressed in terms of single zeta values and \( S(k, l) \), with \( k < r, k + l = r + p \).

4. Other sums. It is interesting to consider also the sum

\[
\begin{align*}
u(p, q, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n + m)^r}
\end{align*}
\]

(4.1)
and compare it to $w(p, q, r)$. Here one can write
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^r + m^r - n^r}{n^{p+r} m^{q+r} (n^r + m^r)} = \zeta(p + r) \zeta(q) - u(p + r, q - r, r).
\] (4.2)

Let $q > p$. We observe that if $(q - p)/r$ is odd, repeating this step $(q - p)/r$ times, we get
\[
u(p, q, r) = \sum_{j=1}^{(q - p)/r} (-1)^j \zeta(p + jr) \zeta(q - (j - 1)r) - u(q, p, r)
\] (4.3)
from where, because of the symmetry $u(p, q, r) = u(q, p, r)$, we obtain the following proposition.

**Proposition 4.1.** For all $q > p \geq 1$, $r \geq 1$ with $(q - p)/r$ odd,
\[
u(p, q, r) = \frac{1}{2} \left(\frac{q - p}{r}\right) \sum_{j=1}^{(q - p)/r} (-1)^{j-1} \zeta(p + jr) \zeta(q - (j - 1)r).
\] (4.4)

Note that $p, q, r$ need not be integers. The only restrictions are those listed above. When $r = 1$ we have
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^p m^q (n + m)} = \frac{1}{2} \sum_{j=1}^{q-1} (-1)^j \zeta(p + j) \zeta(q - j + 1)
\] (4.5)
which can be compared to (2.13). This gives the well-known expression of $S(1, p + q)$ in terms of zeta values. To make this more explicit we set $p = 1$ and $q \geq 2$. Then from (2.13),
\[
w(1, q - 1, 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^{q-1} (n + m)} = S(1, q).
\] (4.6)

This is the same as $u(1, q - 1, 1)$. When $q$ is odd we find from (4.5) (with $p = 1$ and $q - 1$ in the place of $q$)
\[
S(1, q) = \frac{1}{2} \sum_{j=1}^{q-2} (-1)^{j-1} \zeta(j + 1) \zeta(q - j)
\] (4.7)
which is a variant of Euler's formula for the sum $S(1, q)$ (see [5, Theorem 2.2]).

**References**


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