ON $n$-FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

YOUNG BAE JUN

(Received 3 November 2000)

ABSTRACT. We consider the fuzzification of the notion of an $n$-fold implicative ideal, an $n$-fold (weak) commutative ideal. We give characterizations of an $n$-fold fuzzy implicative ideal. We establish an extension property for $n$-fold fuzzy commutative ideals.

2000 Mathematics Subject Classification. 06F35, 03G25, 03E72.

1. Introduction. Huang and Chen [1] introduced the notion of $n$-fold implicative ideals and $n$-fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of $n$-fold implicative ideals, $n$-fold commutative ideals and $n$-fold weak commutative ideals. We show that every $n$-fold fuzzy implicative ideal is an $n$-fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an $n$-fold fuzzy implicative ideal. Using the level set, we provide a characterization of an $n$-fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an $n$-fold fuzzy (weak) commutative ideal. We show that every $n$-fold fuzzy positive implicative ideal which is an $n$-fold fuzzy weak commutative ideal is an $n$-fold fuzzy implicative ideal. Finally, we establish an extension property for $n$-fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a BCK-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

We can define a partial ordering $\leq$ on $X$ by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra $X$, the following hold:

(P1) $x * 0 = x$,
(P2) $x * y \leq x$,
(P3) $(x * y) * z = (x * z) * y$,
(P4) $(x * z) * (y * z) \leq x * y$,
(P5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Throughout, $X$ will always mean a BCK-algebra unless otherwise specified. A non-empty subset $I$ of $X$ is called an ideal of $X$ if it satisfies:

(I) $0 \in I$,
(I2) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \).
A nonempty subset \( I \) of \( X \) is said to be an **implicative ideal** of \( X \) if it satisfies:

(I1) \( 0 \in I \),
(I3) \((x \ast (y \ast x)) \ast z \in I \) and \( z \in I \) imply \( x \in I \).

A nonempty subset \( I \) of \( X \) is said to be a **commutative ideal** of \( X \) if it satisfies:

(I1) \( 0 \in I \),
(I4) \((x \ast y) \ast z \in I \) and \( z \in I \) imply \( x \ast (y \ast (y \ast x)) \in I \).

We now review some fuzzy logic concepts. A fuzzy set in a set \( X \) is a function \( \mu : X \rightarrow [0,1] \).

For a fuzzy set \( \mu \) in \( X \) and \( t \in [0,1] \) define \( U(\mu; t) \) to be the set

\[ U(\mu; t) = \{ x \in X | \mu(x) \geq t \} \] .

A fuzzy set \( \mu \) in \( X \) is said to be a **fuzzy ideal** of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F2) \( \mu(x) \geq \min \{ \mu(x \ast y), \mu(y) \} \) for all \( x, y \in X \).

Note that every fuzzy ideal \( \mu \) of \( X \) is order reversing, that is, if \( x \leq y \) then \( \mu(x) \geq \mu(y) \).

A fuzzy set \( \mu \) in \( X \) is called a **fuzzy implicative ideal** of \( X \) if it satisfies:

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F3) \( \mu(x) \geq \min \{ \mu((x \ast (y \ast x)) \ast z), \mu(z) \} \) for all \( x, y, z \in X \).

A fuzzy set \( \mu \) in \( X \) is called a **fuzzy commutative ideal** of \( X \) if it satisfies:

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F4) \( \mu(x \ast (y \ast (y \ast x))) \geq \min \{ \mu((x \ast y) \ast z), \mu(z) \} \) for all \( x, y, z \in X \).

3. **n-fold fuzzy implicative ideals.** For any elements \( x \) and \( y \) of a BCK-algebra \( X \), \( x \ast y^n \) denotes

\[(\cdots((x \ast y) \ast y) \ast \cdots) \ast y \tag{3.1}\]
in which \( y \) occurs \( n \) times. Huang and Chen [1] introduced the concept of \( n \)-fold implicative ideals as follows.

**Definition 3.1** (see [1]). A subset \( A \) of \( X \) is called an **\( n \)-fold implicative ideal** of \( X \) if

(I1) \( 0 \in A \),
(I5) \((x \ast (y \ast x^n)) \ast z \in A \) and \( z \in A \) imply \( x \in A \) for every \( x, y, z \in X \).

We consider the fuzzification of the concept of \( n \)-fold implicative ideal.

**Definition 3.2.** A fuzzy set \( \mu \) in \( X \) is called an **\( n \)-fold fuzzy implicative ideal** of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F5) \( \mu(x) \geq \min \{ \mu((x \ast (y \ast x^n)) \ast z), \mu(z) \} \) for every \( x, y, z \in X \).

Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

**Theorem 3.3.** Every \( n \)-fold fuzzy implicative ideal is a fuzzy ideal.

**Proof.** The condition (F2) follows from taking \( y = 0 \) in (F5). \( \square \)

The following example shows that the converse of Theorem 3.3 may not be true.
Example 3.4. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, in which the operation $*$ is defined by $x \ast y = \max\{0, x - y\}$ for all $x, y \in X$. Then $X$ is a BCK-algebra (see [1, Example 1.3]). Let $\mu$ be a fuzzy set in $X$ given by $\mu(0) = t_0 > t_1 = \mu(x)$ for all $x(\neq 0) \in X$. Then $\mu$ is a fuzzy ideal of $X$. But $\mu$ is not a 2-fold fuzzy implicative ideal of $X$ because

$$\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 \ast (14 \ast 3^2)) \ast 0), \mu(0)\}. \quad (3.2)$$

We give a condition for a fuzzy ideal to be an $n$-fold fuzzy implicative ideal.

**Theorem 3.5.** A fuzzy ideal $\mu$ of $X$ is $n$-fold fuzzy implicative if and only if $\mu(x) \geq \mu(x \ast (y \ast x^n))$ for all $x, y \in X$.

**Proof.** Necessity is by taking $z = 0$ in (F5). Suppose that a fuzzy ideal $\mu$ satisfies the inequality $\mu(x) \geq \mu(x \ast (y \ast x^n))$ for all $x, y \in X$. Then

$$\mu(x) \geq \mu(x \ast (y \ast x^n)) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\}. \quad (3.3)$$

Hence $\mu$ is an $n$-fold fuzzy implicative ideal of $X$. \hfill \Box

**Theorem 3.6.** A fuzzy set $\mu$ in $X$ is an $n$-fold fuzzy implicative ideal of $X$ if and only if the nonempty level set $\cup\{\mu;x \geq t\}$ of $\mu$ is an $n$-fold implicative ideal of $X$ for every $t \in [0, 1]$.

**Proof.** Assume that $\mu$ is an $n$-fold fuzzy implicative ideal of $X$ and $\cup\{\mu;x \geq t\} \neq \emptyset$ for every $t \in [0, 1]$. Then there exists $x \in \cup\{\mu;x \geq t\}$. It follows from (F1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in \cup\{\mu;x \geq t\}$. Let $x, y, z \in X$ be such that $(x \ast (y \ast x^n)) \ast z \in \cup\{\mu;x \geq t\}$ and $z \in \cup\{\mu;x \geq t\}$. Then $\mu((x \ast (y \ast x^n)) \ast z) \geq t$ and $\mu(z) \geq t$, which imply from (F5) that

$$\mu(x) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\} \geq t \quad (3.4)$$

so that $x \in \cup\{\mu;x \geq t\}$. Therefore $\cup\{\mu;x \geq t\}$ is an $n$-fold implicative ideal of $X$. Conversely, suppose that $\cup\{\mu;x \geq t\} \neq \emptyset$ is an $n$-fold implicative ideal of $X$ for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in \cup\{\mu;x \geq t\}$. Since $0 \notin \cup\{\mu;x \geq t\}$, we get $\mu(0) \geq t = \mu(x)$ and so $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that there exist $a, b, c \in X$ such that

$$\mu(a) < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \quad (3.5)$$

Selecting $s_0 = (1/2)(\mu(a) + \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\})$, then

$$\mu(a) < s_0 < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \quad (3.6)$$

It follows that $(a \ast (b \ast a^n)) \ast c \in \cup\{\mu;x \geq s_0\}$, $c \in \cup\{\mu;x \geq s_0\}$, and $a \notin \cup\{\mu;x \geq s_0\}$. This is a contradiction. Hence $\mu$ is an $n$-fold fuzzy implicative ideal of $X$. \hfill \Box

**Definition 3.7** (see [3]). A fuzzy set $\mu$ in $X$ is called an $n$-fold fuzzy positive implicative ideal of $X$ if

- (F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
- (F6) $\mu(x \ast y^n) \geq \min\{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}$ for all $x, y, z \in X$. 

Lemma 3.8 (see [3, Theorem 3.13]). Let $\mu$ be a fuzzy set in $X$. Then $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if the nonempty level set $U(\mu; t)$ of $\mu$ is an $n$-fold positive implicative ideal of $X$ for every $t \in [0, 1]$.

Lemma 3.9 (see [1, Theorem 2.5]). Every $n$-fold implicative ideal is an $n$-fold positive implicative ideal.

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

Theorem 3.10. Every $n$-fold fuzzy implicative ideal is an $n$-fold fuzzy positive implicative ideal.

4. $n$-fold fuzzy commutative ideals

Definition 4.1 (see [1]). A subset $A$ of $X$ is called an $n$-fold commutative ideal of $X$ if

(I1) $0 \in A$,
(I6) $(x \ast y) \ast z \in A$ and $z \in A$ imply $x \ast (y \ast (y \ast x^n)) \in A$ for all $x, y, z \in X$.

A subset $A$ of $X$ is called an $n$-fold weak commutative ideal of $X$ if

(I1) $0 \in A$,
(I7) $(x \ast (x \ast y^n)) \ast z \in A$ and $z \in A$ imply $y \ast (y \ast x) \in A$ for all $x, y, z \in X$.

We consider the fuzzification of $n$-fold (weak) commutative ideals as follows.

Definition 4.2. A fuzzy set $\mu$ in $X$ is called an $n$-fold fuzzy commutative ideal of $X$ if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(F7) $\mu(x \ast (y \ast (y \ast x^n))) \geq \min\{\mu((x \ast y) \ast z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set $\mu$ in $X$ is called an $n$-fold fuzzy weak commutative ideal of $X$ if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(F8) $\mu(y \ast (y \ast x)) \geq \min\{\mu((x \ast (x \ast y^n)) \ast z), \mu(z)\}$ for all $x, y, z \in X$.

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting $y = 0$ and $y = x$ in (F7) and (F8), respectively, we know that every $n$-fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

Theorem 4.3. Let $\mu$ be a fuzzy ideal of $X$. Then

(i) $\mu$ is an $n$-fold fuzzy commutative ideal of $X$ if and only if

$$\mu(x \ast (y \ast (y \ast x^n))) \geq \mu(x \ast y) \quad \forall x, y \in X.$$  \hspace{1cm} (4.1)

(ii) $\mu$ is an $n$-fold fuzzy weak commutative ideal of $X$ if and only if

$$\mu(y \ast (y \ast x)) \geq \mu(x \ast (x \ast y^n)) \quad \forall x, y \in X.$$  \hspace{1cm} (4.2)

Proof. The proof is straightforward.

Lemma 4.4 (see [3, Theorem 3.12]). A fuzzy set $\mu$ in $X$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if $\mu$ is a fuzzy ideal of $X$ in which the following inequality holds:

(F9) $\mu((x \ast z^n) \ast (y \ast z^n)) \geq \mu((x \ast y) \ast z^n) \quad \forall x, y, z \in X$. 
**Theorem 4.5.** If $\mu$ is both an $n$-fold fuzzy positive implicative ideal and an $n$-fold fuzzy weak commutative ideal of $X$, then it is an $n$-fold fuzzy implicative ideal of $X$.

**Proof.** Let $x, y \in X$. Using **Theorem 4.3**(ii), **Lemma 4.4**, (P3), and (III), we have

\[
\begin{align*}
\mu(x \ast (x \ast (y \ast x^n))) & \geq \mu((y \ast x^n) \ast ((y \ast x^n) \ast x^n)) \\
& \geq \mu((y \ast (y \ast x^n)) \ast x^n) \\
& = \mu((y \ast x^n) \ast (y \ast x^n)) \quad (4.3) \\
& = \mu(0).
\end{align*}
\]

It follows from (F1) and (F2) that

\[
\begin{align*}
\mu(x) & \geq \min \{\mu(x \ast (x \ast (y \ast x^n))), \mu(x \ast (y \ast x^n))\} \\
& \geq \min \{\mu(0), \mu(x \ast (y \ast x^n))\} \quad (4.4) \\
& = \mu(x \ast (y \ast x^n))
\end{align*}
\]

so from **Theorem 3.5**, $\mu$ is an $n$-fold fuzzy implicative ideal of $X$.

**Theorem 4.6** (extension property for $n$-fold fuzzy commutative ideals). Let $\mu$ and $\nu$ be fuzzy ideals of $X$ such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If $\mu$ is an $n$-fold fuzzy commutative ideal of $X$, then so is $\nu$.

**Proof.** Let $x, y \in X$. Taking $u = x \ast (x \ast y)$, we have

\[
\begin{align*}
\nu(0) = \mu(0) & = \mu(u \ast y) \\
& \leq \mu(u \ast (y \ast (y \ast u^n))) \\
& \leq \nu(u \ast (y \ast (y \ast u^n))) \quad (4.5) \\
& = \nu((x \ast (x \ast y)) \ast (y \ast (y \ast u^n))) \\
& = \nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)).
\end{align*}
\]

Since $x \ast (y \ast (y \ast u^n)) \leq x \ast (y \ast (y \ast u^n))$ and since $\nu$ is order reversing, it follows that

\[
\begin{align*}
\nu(x \ast (y \ast (y \ast x^n))) & \geq \nu(x \ast (y \ast (y \ast u^n))) \\
& \geq \min \{\nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)), \nu(x \ast y)\} \quad (4.6) \\
& \geq \min \{\nu(0), \nu(x \ast y)\} \\
& = \nu(x \ast y).
\end{align*}
\]

Hence, by **Theorem 4.3**(i), $\nu$ is an $n$-fold fuzzy commutative ideal of $X$.

**Acknowledgement.** This work was supported by Korea Research Foundation Grant (KRF-2000-005-D00003).

**References**


Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: ybjun@nongae.gsnu.ac.kr