ISHIKAWA ITERATION PROCESS WITH ERRORS FOR NONEXPANSIVE MAPPINGS

JIALIN HUANG

(Received 16 June 2000)

ABSTRACT. We study the construction and the convergence of the Ishikawa iterative process with errors for nonexpansive mappings in uniformly convex Banach spaces. Some recent corresponding results are generalized.

2000 Mathematics Subject Classification. 47H10, 40A05.

1. Introduction. Let $C$ be a closed convex subset of a Banach space $X$ and $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$). Recently, Deng and Li [1] introduced an Ishikawa iteration sequence with errors as follows: for any given $x_0 \in C$

$$
x_{n+1} = \alpha_n x_n + \beta_n Ty_n + y_n u_n, \quad y_n = \hat{\alpha}_n x_n + \hat{\beta}_n Tx_n + \hat{y}_n v_n, \quad n \geq 0. \quad (1.1)
$$

Here $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in $C$, and $\{\alpha_n\}, \{\beta_n\}, \{y_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\},$ and $\{\hat{y}_n\}$ are six sequences in $[0,1]$ satisfying the conditions

$$
\alpha_n + \beta_n + y_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{y}_n = 1 \quad \forall n \geq 0. \quad (1.2)
$$

Remark 1.1. Note that the Ishikawa iteration processes [2] is a special case of the Ishikawa iteration processes with errors.

Deng and Li [1] obtained the following result. Let $C$ be a closed convex subset of a uniformly convex Banach space $X$. If for any initial guess $x_0 \in C$, $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n \beta_n \hat{\beta}_n < \infty$, $\sum_{n=0}^{\infty} y_n < \infty$, and $\sum_{n=0}^{\infty} \hat{y}_n < \infty$, then $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. So Deng and Li extended the result of Tan and Xu [6].

In this paper, we first extend and unify [1, Theorem 1] and [6, Lemma 3]. Then, we generalize [1, Theorems 2, 3, and 4] and [6, Theorems 1, 2, and 3].

2. Lemmas

Lemma 2.1 (see [6]). Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n \to \infty} a_n$ exists.
**Lemma 2.2** (see [1]). Let $C$ be a closed convex subset of a Banach space $X$, $T : C \to C$ a nonexpansive mapping. Then for any initial guess $x_0 \in C$, \{x_n\} defined by (1.1),

$$
||x_{n+1} - p|| \leq ||x_n - p|| + y_n\|u_n - p\| + \beta_n\|v_n - p\|
$$

(2.1)

for all $n \geq 1$ and for all $p \in F(T)$, where $F(T)$ denotes the set of fixed point of $T$.

**Remark 2.3.** Since the sequences \{u_n\} and \{v_n\} are bounded, so the sequences \{\|u_n - p\|\} and \{\|v_n - p\|\} are bounded too, then $\liminf_{n \to \infty} \{\|x_n - p\|\}$ exists by Lemma 2.1.

**Lemma 2.4** (see [7]). Let $C$ be a closed convex subset of a Banach space $X$. Suppose that $T : C \to C$ is a nonexpansive mapping. If $y_n \to y$ weakly ($y_n, y \in C, n = 1, 2, \ldots$), then there exists a strictly increasing convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$
g(\|y - Ty\|) \leq \liminf_{n \to \infty} \|y_n - Ty_n\|.
$$

(2.2)

3. Main results

**Theorem 3.1.** Let $C$ be a closed convex subset of a uniformly convex Banach space $X$, $T : C \to C$ a nonexpansive mapping with a fixed point. If for any initial guess $x_0 \in C$, \{x_n\} defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} y_n < \infty$, $\sum_{n=0}^{\infty} \hat{y}_n < \infty$, and there exists a subsequence \{n_k\} of \{n\} such that $\sum_{k=0}^{\infty} \alpha_k \beta_{nk} = \infty$, $\sum_{k=0}^{\infty} \alpha_k \hat{\beta}_{nk} < \infty$. Then $\liminf_{n \to \infty} \|x_n - Tx_n\| = 0$.

**Proof.** Since $T$ has a fixed point, and by Lemma 2.2, we may set

$$
M = \sup_{n \geq 0} \{\|Tx_n - u_n\|, \|x_n - u_n\|, \|Ty_n - v_n\|, \|y_n - u_n\|, \|x_n - u_n\|\}. \quad (3.1)
$$

If $\liminf_{n \to \infty} \|x_n - Tx_n\| > 0$, we may assume that $\liminf_{n \to \infty} \|x_n - p\| > 0$, where $p \in F(T)$. Since $\|Ty_n - p\| \leq \|x_n - p\| + \hat{y}_n M$, we obtain

$$
\|x_{n+1} - p\| \leq \|\alpha_n (x_n - p) + \beta_n (Ty_n - p)\| + y_n M
$$

$$
= (\alpha_n + \beta_n) \left\| \frac{\alpha_n}{\alpha_n + \beta_n} (x_n - p) + \frac{\beta_n}{\alpha_n + \beta_n} (Ty_n - p) \right\| + y_n M
$$

$$
\leq \left[ 1 - 2 \frac{\alpha_n \beta_n}{(\alpha_n + \beta_n)^2} \right] \delta_X \left( \frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{y}_n M} \right) \left( \|x_n - p\| + \hat{y}_n M \right) + y_n M \quad (3.2)
$$

$$
\leq \left[ 1 - 2 \alpha_n \beta_n \delta_X \left( \frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{y}_n M} \right) \right] \|x_n - p\| + (\hat{y}_n + y_n) M,
$$

where $\delta_X$ is the modulus of convexity of the uniformly convex Banach space $X$. Setting

$$
D_n = 1 - 2 \alpha_n \beta_n \delta_X \left( \frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{y}_n M} \right).
$$

(3.3)
Thus for all $n \geq 0$, $0 \leq D_n \leq 1$. From (3.2), for all $k \geq 0$, we have

\[
\|x_{n_{k+1}} - p\| \\
\leq D_{n_{k+1}} \|x_{n_{k+1} - 1} - p\| + (\hat{y}_{n_{k+1} - 1} + y_{n_{k+1} - 1}) M \\
\leq D_{n_{k+1}} D_{n_{k+1} - 2} \cdots D_{n_k} \|x_n - p\| + \sum_{i=1}^{n_{k+1} - n_k} (\hat{y}_{n_{k+1} - i} + y_{n_{k+1} - i}) M \\
\leq D_n \|x_n - p\| + \sum_{i=1}^{n_{k+1} - n_k} (\hat{y}_{n_{k+1} - i} + y_{n_{k+1} - i}) M
\]

(3.4)

Thus, 

\[
\sum_{i=0}^{k} \left[ 2 \alpha_{n_i} \beta_{n_i} \delta \left( \frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) \right] \|x_{n_i} - p\| \\
\leq \|x_n - p\| - \|x_{n_{k+1}} - p\| + \sum_{i=1}^{n_{k+1} - n_k} (\hat{y}_{n_{k+1} - i} + y_{n_{k+1} - i}) M.
\]

(3.5)

It follows that 

\[
\sum_{i=0}^{\infty} \left[ \alpha_{n_i} \beta_{n_i} \delta \left( \frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) \right] < +\infty.
\]

(3.6)

By condition $\sum_{i=0}^{\infty} \alpha_{n_i} \beta_{n_i} \hat{\beta}_{n_i} < +\infty$, we have 

\[
\sum_{i=0}^{\infty} \left[ \alpha_{n_i} \beta_{n_i} \left[ \delta \left( \frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) + \hat{\beta}_{n_i} \right] \right] < +\infty.
\]

(3.7)

It follows that 

\[
\liminf_{k \to \infty} \left[ \delta \left( \frac{\|x_{n_k} - Ty_{n_k}\|}{\|x_{n_k} - p\| + \hat{y}_{n_k} M} \right) + \hat{\beta}_{n_k} \right] = 0
\]

(3.8)

since $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} = \infty$. Hence, there is a sequence $\{n_k\} \subset \{n_k\}$ such that 

\[
\lim_{i \to \infty} \|x_{n_{k_i}} - Ty_{n_{k_i}}\| = 0, \quad \lim_{i \to \infty} \hat{\beta}_{n_{k_i}} = 0.
\]

(3.9)

On the other hand, we have 

\[
\|x_{n_{k_i}} - Tx_{n_{k_i}}\| \leq \|x_{n_{k_i}} - Ty_{n_{k_i}}\| + \|Tx_{n_{k_i}} - Ty_{n_{k_i}}\| \\
\leq \|x_{n_{k_i}} - Ty_{n_{k_i}}\| + \hat{\beta}_{n_{k_i}} \|x_{n_{k_i}} - Tx_{n_{k_i}}\| + \hat{y}_{n_{k_i}} M.
\]

(3.10)

Setting $i \to \infty$ in (3.10), it follows from (3.9) that 

\[
\lim_{i \to \infty} \|x_{n_{k_i}} - Tx_{n_{k_i}}\| = 0.
\]

(3.11)
Thus,
\[ \liminf_{n \to \infty} \|x_n - Tx_n\| = 0. \] (3.12)

This completes the proof. \(\square\)

Recall that a Banach space \(X\) is said to satisfy Opial’s condition [4] if the condition \(x_n \to x_0\) weakly implies
\[ \limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - y\| \quad \forall y \neq x_0. \] (3.13)

A mapping \(T : C \to C\) with a nonempty fixed points set \(F(T)\) in \(C\) will be said to satisfy Condition A in [5] if there is a nondecreasing function \(f : (0, \infty) \to (0, \infty)\) with \(f(0) = 0, f(r) > 0\) for \(r \in (0, \infty)\), such that \(\|x - Tx\| \geq f(d(x, F(T)))\) for all \(x \in C\), where \(d(x, F(T)) = \inf \{\|x - z\| : z \in F(T)\}\).

**THEOREM 3.2.** Let \(C\) be a bounded closed convex subset of a uniformly convex Banach space \(X\) which satisfies Opial’s condition or whose norm is Fréchet differentiable. Let \(T : C \to C\) a nonexpansive mapping with a fixed point, and \(\{x_n\}\) defined by (1.1), with the restrictions that \(\sum_{n=0}^{\infty} y_n < \infty, \sum_{n=0}^{\infty} y_n < \infty, \) and for any subsequence \(\{n_k\}\) of \(\{n\}\), \(\sum_{k=0}^{\infty} \alpha_n \beta^k < \infty, \sum_{k=0}^{\infty} \alpha_n \beta_n \beta^k < \infty,\) converges weakly to a fixed point of \(T\).

By Theorem 3.1 and Lemma 2.4, we can prove Theorem 3.2 easily. The proof is similar to that of [7, Theorem 3.1], so the details are omitted.

Let \(X, C, T, \) and \(\{x_n\}\) be as in Theorem 3.1. Then we have the following theorem.

**THEOREM 3.3.** If the range of \(C\) under \(T\) is contained in a compact subset of \(X\), then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**THEOREM 3.4.** Let \(C\) be a bounded closed convex subset of a uniformly convex Banach space \(X\). If \(T\) satisfies Condition A, then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**Proof.** Since \(C\) is a bounded closed convex subset of a uniformly convex Banach space \(X\), then \(T\) has a fixed point [3]. So \(F(T)\) is nonempty. It follows from Theorem 3.1 and Condition A, that there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\lim_{k \to \infty} f(d(x_{n_k}, F(T))) = 0\), therefore we have \(\lim_{k \to \infty} d(x_{n_k}, F(T)) = 0\). So we can choose a subsequence \(\{x_{n_{k_i}}\}\) of \(\{x_{n_k}\}\) and some sequence \(\{p_i\}\) in \(F(T)\) such that \(\|x_{n_{k_i}} - p_i\| < 2^{-i}\) for all integers \(k \geq 0\).

We denote \(\sup\{\|u_n - p_i\|, \|v_n - p_i\|\}\) by \(M\) and \((y_{n_{k_i}} + \beta_{n_{k_i}} \tilde{y}_{n_{k_i}})M\) by \(\lambda_{n_{k_i}}\). By Lemma 2.1 we have
\[
\|p_{i+1} - p_i\| \leq \|x_{n_{k_{i+1}}} - p_{i+1}\| + \|x_{n_{k_{i+1}}} - p_i\| \\
\leq 2^{-i+1} + \|x_{n_{k_{i+1}}} - p_i\| + \lambda_{n_{k_{i+1}} - 1} \\
\leq 2^{-i+1} + \|x_{n_{k_{i+1}}} - 2 - p_i\| + \lambda_{n_{k_{i+1}} - 2} + \lambda_{n_{k_{i+1}} - 1}
\]
ISHIKAWA ITERATION PROCESS WITH ERRORS …

\[ \leq 2^{-(i+1)} + \left\| x_{n_{ki}} - p_i \right\| + \sum_{j=n_{ki}}^{n_{ki+1}-1} \lambda_j \]

\[ \leq 2^{-(i+1)} + \sum_{j=n_{ki}}^{n_{ki+1}-1} \lambda_j, \]

(3.14)

It follows, from (3.14) and \( \sum_j \lambda_j \) is convergent, that \( \{p_i\} \) is a Cauchy sequence therefore converges strongly to a point \( p \in F(T) \), since \( F(T) \) is closed. We have seen that \( \{x_{n_{ki}}\} \) converges strongly to \( p \), so does \( \{x_n\} \) by the Remark 2.3. This completes the proof.

**Remark 3.5.** The above three theorems generalize [6, Theorems 1, 2, and 3] and [1, Theorems 2, 3, and 4], respectively.

**References**


JIALIN HUANG: DEPARTMENT OF MATHEMATICS, YIBIN TEACHER’S COLLEGE, YIBIN 644007, CHINA

E-mail address: jlhuang@btamail.net.cn