ON IMAGINABLE $T$-FUZZY SUBALGEBRAS AND IMAGINABLE $T$-FUZZY CLOSED IDEALS IN BCH-ALGEBRAS

YOUNG BAE JUN and SUNG MIN HONG

(Received 9 December 2000)

ABSTRACT. We inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

2000 Mathematics Subject Classification. 06F35, 03G25, 94D05.

1. Introduction. In 1983, Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [6, 7]). In [4], Chaudhry et al. stated ideals and filters in BCH-algebras, and studied their properties. For further properties on BCH-algebras, we refer to [2, 3, 5]. In [8], the first author considered the fuzzification of ideals and filters in BCH-algebras, and then described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters in BCH-algebras. In this paper, we inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

2. Preliminaries. By a $BCH$-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

- (H1) $x * x = 0$,
- (H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
- (H3) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.

In a $BCH$-algebra $X$, the following statements hold:

- (P1) $x * 0 = x$. 

(P2) \( x \ast 0 = 0 \) implies \( x = 0 \).

(P3) \( 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) \).

A nonempty subset \( A \) of a BCH-algebra \( X \) is called a subalgebra of \( X \) if \( x \ast y \in A \) whenever \( x, y \in A \). A nonempty subset \( A \) of a BCH-algebra \( X \) is called a closed ideal of \( X \) if

(i) \( 0 \ast x \in A \) for all \( x \in A \),

(ii) \( x \ast y \in A \) and \( y \in A \) imply that \( x \in A \).

In what follows, let \( X \) denote a BCH-algebra unless otherwise specified. A fuzzy set in \( X \) is a function \( \mu : X \to [0,1] \). Let \( \mu \) be a fuzzy set in \( X \). For \( \alpha \in [0,1] \), the set \( U(\mu;\alpha) = \{ x \in X \mid \mu(x) \geq \alpha \} \) is called a level set of \( \mu \).

A fuzzy set \( \mu \) in \( X \) is called a fuzzy subalgebra of \( X \) if

\[
\mu(x \ast y) \geq \min \{ \mu(x), \mu(y) \}, \quad \forall x, y \in X.
\] (2.1)

**Definition 2.1** (see [1]). By a \( t \)-norm \( T \) on \([0,1]\), we mean a function \( T : [0,1] \times [0,1] \to [0,1] \) satisfying the following conditions:

(T1) \( T(x,1) = x \),

(T2) \( T(x,y) \leq T(x,z) \) if \( y \leq z \),

(T3) \( T(x,y) = T(y,x) \),

(T4) \( T(x,T(y,z)) = T(T(x,y),z) \), for all \( x, y, z \in [0,1] \).

In what follows, let \( T \) denote a \( t \)-norm on \([0,1]\) unless otherwise specified. Denote by \( \Delta_T \) the set of elements \( \alpha \in [0,1] \) such that \( T(\alpha,\alpha) = \alpha \), that is,

\[
\Delta_T := \{ \alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha \}.
\] (2.2)

Note that every \( t \)-norm \( T \) has a useful property:

(P4) \( T(\alpha,\beta) \leq \min(\alpha,\beta) \) for all \( \alpha, \beta \in [0,1] \).

3. Fuzzy closed ideals

**Definition 3.1** (see [8]). A fuzzy set \( \mu \) in \( X \) is called a fuzzy closed ideal of \( X \) if

(F1) \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \),

(F2) \( \mu(x) \geq \min\{\mu(x \ast y),\mu(y)\} \) for all \( x, y \in X \).

**Theorem 3.2.** Let \( D \) be a subset of \( X \) and let \( \mu_D \) be a fuzzy set in \( X \) defined by

\[
\mu_D(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D, \\
\alpha_2 & \text{if } x \notin D,
\end{cases}
\] (3.1)

for all \( x \in X \) and \( \alpha_1 > \alpha_2 \). Then \( \mu_D \) is a fuzzy closed ideal of \( X \) if and only if \( D \) is a closed ideal of \( X \).

**Proof.** Assume that \( \mu_D \) is a fuzzy closed ideal of \( X \). Let \( x \in D \). Then, by (F1), we have \( \mu(0 \ast x) \geq \mu(x) = \alpha_1 \) and so \( \mu(0 \ast x) = \alpha_1 \). It follows that \( 0 \ast x \in D \). Let \( x, y \in X \) be such that \( x \ast y \in D \) and \( y \in D \). Then \( \mu_D(x \ast y) = \alpha_1 = \mu_D(y) \), and hence

\[
\mu_D(x) \geq \min\{\mu_D(x \ast y),\mu_D(y)\} = \alpha_1.
\] (3.2)

Thus \( \mu_D(x) = \alpha_1 \), that is, \( x \in D \). Therefore \( D \) is a closed ideal of \( X \).
Conversely, suppose that $D$ is a closed ideal of $X$. Let $x \in X$. If $x \notin D$, then $0 \ast x \notin D$ and thus $\mu_D(0 \ast x) = \alpha_1 = \mu_D(x)$. If $x \in D$, then $\mu_D(x) = \alpha_2 \leq \mu_D(0 \ast x)$. Let $x, y \in X$. If $x \ast y \in D$ and $y \in D$, then $x \in D$. Hence

$$
\mu_D(x) = \alpha_1 = \min \{ \mu_D(x \ast y), \mu_D(y) \}. \tag{3.3}
$$

If $x \ast y \notin D$ and $y \notin D$, then clearly $\mu_D(x) \geq \min \{ \mu_D(x \ast y), \mu_D(y) \}$. If exactly one of $x \ast y$ and $y$ belong to $D$, then exactly one of $\mu_D(x \ast y)$ and $\mu_D(y)$ is equal to $\alpha_2$. Therefore, $\mu_D(x) \geq \alpha_2 = \min \{ \mu_D(x \ast y), \mu_D(y) \}$. Consequently, $\mu_D$ is a fuzzy closed ideal of $X$.

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

**Theorem 3.3.** A fuzzy set $\mu$ in $X$ is a fuzzy closed ideal of $X$ if and only if the nonempty level set $U(\mu; \alpha)$ of $\mu$ is a closed ideal of $X$ for all $\alpha \in [0, 1]$.

We then call $U(\mu; \alpha)$ a level closed ideal of $\mu$.

**Proof.** Assume that $\mu$ is a fuzzy closed ideal of $X$ and $U(\mu; \alpha) \neq \emptyset$ for all $\alpha \in [0, 1]$. Let $x \in U(\mu; \alpha)$. Then $\mu(0 \ast x) = \mu(x) \geq \alpha$, and so $0 \ast x \in U(\mu; \alpha)$. Let $x, y \in X$ be such that $x \ast y \in U(\mu; \alpha)$ and $y \in U(\mu; \alpha)$. Then

$$
\mu(x) \geq \min \{ \mu(x \ast y), \mu(y) \} \geq \min \{ \alpha, \alpha \} = \alpha, \tag{3.4}
$$

and thus $x \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is a closed ideal of $X$. Conversely, suppose that $U(\mu; \alpha) \neq \emptyset$ is a closed ideal of $X$. If $\mu(0 \ast a) < \mu(a)$ for some $a \in X$, then $\mu(0 \ast a) < \alpha_0 < \mu(a)$ by taking $\alpha_0 := 1/2(\mu(0 \ast a) + \mu(a))$. It follows that $a \in U(\mu; \alpha_0)$ and $0 \ast a \notin U(\mu; \alpha_0)$, which is a contradiction. Hence $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. Assume that there exist $x_0, y_0 \in X$ such that

$$
\mu(x_0) < \min \{ \mu(x_0 \ast y_0), \mu(y_0) \}. \tag{3.5}
$$

Taking $\beta_0 := 1/2(\mu(x_0) + \min \{ \mu(x_0 \ast y_0), \mu(y_0) \})$, we get $\mu(x_0) < \beta_0 < \mu(x_0 \ast y_0)$ and $\mu(x_0) < \beta_0 < \mu(y_0)$. Thus $x_0 \ast y_0 \in U(\mu; \beta_0)$ and $y_0 \in U(\mu; \beta_0)$, but $x_0 \notin U(\mu; \beta_0)$. This is impossible. Hence $\mu$ is a fuzzy closed ideal of $X$. \qed

**Theorem 3.4.** Let $\mu$ be a fuzzy set in $X$ and $\text{Im}(\mu) = \{ \alpha_0, \alpha_1, \ldots, \alpha_n \}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{ D_k \mid k = 0, 1, 2, \ldots, n \}$ be a family of closed ideals of $X$ such that

(i) $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = X$,

(ii) $\mu(D^*_k) = \alpha_k$, where $D^*_k = D_k \setminus D_{k-1}$ and $D_{-1} = \emptyset$ for $k = 0, 1, \ldots, n$.

Then $\mu$ is a fuzzy closed ideal of $X$.

**Proof.** For any $x \in X$ there exists $k \in \{0, 1, \ldots, n\}$ such that $x \in D^*_k$. Since $D_k$ is a closed ideal of $X$, it follows that $0 \ast x \in D_k$. Thus $\mu(0 \ast x) \geq \alpha_k = \mu(x)$. To prove that $\mu$ satisfies condition (F2), we discuss the following cases: if $x \ast y \in D^*_k$ and $y \in D^*_k$, then $x \in D_k$ because $D_k$ is a closed ideal of $X$. Hence

$$
\mu(x) \geq \alpha_k = \min \{ \mu(x \ast y), \mu(y) \}. \tag{3.6}
$$
If \( x \ast y \notin D^*_k \) and \( y \notin D^*_k \), then the following four cases arise:

1. \( x \ast y \in X \backslash D_k \) and \( y \in X \backslash D_k \),
2. \( x \ast y \in D_{k-1} \) and \( y \in D_{k-1} \),
3. \( x \ast y \in X \backslash D_k \) and \( y \in D_{k-1} \),
4. \( x \ast y \in D_{k-1} \) and \( y \in X \backslash D_k \).

But, in either case, we know that \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \). If \( x \ast y \in D^*_k \) and \( y \notin D^*_k \), then either \( y \in D_{k-1} \) or \( y \in X \backslash D_k \). It follows that either \( x \in D_k \) or \( x \in X \backslash D_k \). Thus \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \). Similarly for the case \( x \ast y \notin D^*_k \) and \( y \in D^*_k \), we have the same result. This completes the proof.

**Theorem 3.5.** Let \( \Lambda \) be a subset of \([0, 1]\) and let \( \{D_\lambda \mid \lambda \in \Lambda\} \) be a collection of closed ideals of \( X \) such that

1. \( X = \bigcup_{\lambda \in \Lambda} D_\lambda \).
2. \( \alpha > \beta \) if and only if \( D_\alpha \subseteq D_\beta \) for all \( \alpha, \beta \in \Lambda \).

Define a fuzzy set \( \mu \) in \( X \) by \( \mu(x) = \sup\{\lambda \in \Lambda \mid x \in D_\lambda\} \) for all \( x \in X \). Then \( \mu \) is a fuzzy closed ideal of \( X \).

**Proof.** Let \( x \in X \). Then there exists \( \alpha_i \in \Lambda \) such that \( x \in D_{\alpha_i} \). It follows that \( 0 \ast x = D_{\alpha_j} \) for some \( \alpha_j \geq \alpha_i \). Hence

\[
\mu(x) = \sup\{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_i\} \leq \sup\{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_j\} = \mu(0 \ast x). \tag{3.7}
\]

Let \( x, y \in X \) be such that \( \mu(x \ast y) = m \) and \( \mu(y) = n \), where \( m, n \in [0, 1] \). Without loss of generality we may assume that \( m \leq n \). To prove \( \mu \) satisfies condition (F2), we consider the following three cases:

\[
\begin{align*}
(1^*) \lambda & \leq m, \\
(2^*) m & < \lambda \leq n, \\
(3^*) \lambda & > n.
\end{align*} \tag{3.8}
\]

Case \((1^*)\) implies that \( x \ast y \in D_\lambda \) and \( y \in D_\lambda \). It follows that \( x \in D_\lambda \) so that

\[
\mu(x) = \sup\{\lambda \in \Lambda \mid x \in D_\lambda\} \geq m = \min\{\mu(x \ast y), \mu(y)\}. \tag{3.9}
\]

For the case \((2^*)\), we have \( x \ast y \notin D_\lambda \) and \( y \in D_\lambda \). Then either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \), then \( \mu(x) = n \geq \min\{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \), then \( x \in D_{\delta - \lambda} \) for some \( \delta < \lambda \), and so \( \mu(x) \geq m = \min\{\mu(x \ast y), \mu(y)\} \). Finally, case \((3^*)\) implies \( x \ast y \notin D_\lambda \) and \( y \notin D_\lambda \). Thus we have that either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \) then obviously \( \mu(x) \geq m = \min\{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \) then \( x \in D_{\epsilon - \lambda} \) for some \( \epsilon < \lambda \), and thus \( \mu(x) \geq m = \min\{\mu(x \ast y), \mu(y)\} \). This completes the proof. \( \square \)

Let \( D \) be a subset of \( X \). The least closed ideal of \( X \) containing \( D \) is called the closed ideal generated by \( D \), denoted by \( \langle D \rangle \). Note that if \( C \) and \( D \) are subsets of \( X \) and \( C \subseteq D \), then \( \langle C \rangle \subseteq \langle D \rangle \). Let \( \mu \) be a fuzzy set in \( X \). The least fuzzy closed ideal of \( X \) containing \( \mu \) is called a fuzzy closed ideal of \( X \) generated by \( \mu \), denoted by \( \langle \mu \rangle \).

**Lemma 3.6.** For a fuzzy set \( \mu \) in \( X \), then

\[
\mu(x) = \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}, \quad \forall x \in X. \tag{3.10}
\]

**Proof.** Let \( \delta := \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \) and let \( \varepsilon > 0 \) be given. Then \( \delta - \varepsilon < \alpha \) for some \( \alpha \in [0, 1] \) such that \( x \in U(\mu; \alpha) \), and so \( \delta - \varepsilon < \mu(x) \). Since \( \varepsilon \) is arbitrary, it
follows that \( \mu(x) \geq \delta \). Now let \( \mu(x) = \beta \). Then \( x \in U(\mu; \beta) \) and hence \( \beta \in \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \} \). Therefore
\[
\mu(x) = \beta \leq \sup \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \} = \delta,
\]
and consequently \( \mu(x) = \delta \), as desired. \( \square \)

**Theorem 3.7.** Let \( \mu \) be a fuzzy set in \( X \). Then the fuzzy set \( \mu^* \) in \( X \) defined by
\[
\mu^*(x) = \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}
\]
for all \( x \in X \) is the fuzzy closed ideal \( (\mu) \) generated by \( \mu \).

**Proof.** We first show that \( \mu^* \) is a fuzzy closed ideal of \( X \). For any \( y \in \text{Im}(\mu^*) \), let \( y_n = y - 1/n \) for any \( n \in \mathbb{N} \), where \( \mathbb{N} \) is the set of all positive integers, and let \( x \in U(\mu^*; y) \). Then \( \mu^*(x) \geq y \), and so
\[
\sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} \geq y > y_n,
\]
for all \( n \in \mathbb{N} \). Hence there exists \( \beta \in [0,1] \) such that \( \beta > y_n \) and \( x \in \langle U(\mu; \beta) \rangle \). It follows that \( U(\mu; \beta) \subseteq U(\mu; y_n) \) so that \( x \in \langle U(\mu; \beta) \rangle \subseteq \langle U(\mu; y_n) \rangle \) for all \( n \in \mathbb{N} \). Consequently, \( x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle \). On the other hand, if \( x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle \), then \( y_n \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} \) for any \( n \in \mathbb{N} \). Therefore
\[
y - \frac{1}{n} = y_n \leq \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} = \mu^*(x),
\]
for all \( n \in \mathbb{N} \). Since \( n \) is an arbitrary positive integer, it follows that \( y \leq \mu^*(x) \) so that \( x \in U(\mu^*; y) \). Hence \( U(\mu^*; y) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle \), which is a closed ideal of \( X \). Using Theorem 3.3, we know that \( \mu^* \) is a fuzzy closed ideal of \( X \). We now prove that \( \mu^* \) contains \( \mu \). For any \( x \in X \), let \( \beta \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} \). Then \( x \in U(\mu; \beta) \) and so \( x \in \langle U(\mu; \beta) \rangle \). Thus we get \( \beta \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} \), and so
\[
\{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \} \subseteq \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}.
\]
It follows from Lemma 3.6 that
\[
\mu(x) = \sup \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \}
\leq \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}
= \mu^*(x).
\]
Hence \( \mu \subseteq \mu^* \). Finally let \( v \) be a fuzzy closed ideal of \( X \) containing \( \mu \) and let \( x \in X \). If \( \mu^*(x) = 0 \), then clearly \( \mu^*(x) \leq v(x) \). Assume that \( \mu^*(x) = y \neq 0 \). Then \( x \in U(\mu^*; y) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle \), that is, \( x \in U(\mu; y_n) \) for all \( n \in \mathbb{N} \). It follows that \( v(x) \geq \mu(x) \geq y_n = y - 1/n \) for all \( n \in \mathbb{N} \) so that \( v(x) \geq y = \mu^*(x) \) since \( n \) is arbitrary. This shows that \( \mu^* \leq \mu \), completing the proof. \( \square \)

**Definition 3.8.** A fuzzy closed ideal \( \mu \) of \( X \) is said to be \( n \)-valued if \( \text{Im}(\mu) \) is a finite set of \( n \) elements. When no specific \( n \) is intended, we call \( \mu \) a finite-valued fuzzy closed ideal.
Theorem 3.9. Let \( \mu \) be a fuzzy closed ideal of \( X \). Then \( \mu \) is finite valued if and only if there exists a finite-valued fuzzy set \( \nu \) in \( X \) which generates \( \mu \). In this case, the range sets of \( \mu \) and \( \nu \) are identical.

Proof. If \( \mu : X \to [0,1] \) is a finite-valued fuzzy closed ideal of \( X \), then we may choose \( \nu = \mu \). Conversely, assume that \( \nu : X \to [0,1] \) is a finite-valued fuzzy set. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be distinct elements of \( \nu(X) \) such that \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \), and let 
\[ C_i = \nu^{-1}(\alpha_i) \text{ for } i = 1,2,\ldots,n. \]
Clearly, \( \cup_{i=1}^j C_i \subseteq \cup_{i=1}^k C_i \) whenever \( j < k \leq n \). Hence if we let \( D_j = \langle \cup_{i=1}^j C_i \rangle \), then we have the following chain:
\[
D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n = X. \quad (3.17)
\]
Define a fuzzy set \( \mu : X \to [0,1] \) as follows:
\[
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D_1, \\
\alpha_j & \text{if } x \in D_j \setminus D_{j-1}.
\end{cases} \quad (3.18)
\]
We claim that \( \mu \) is a fuzzy closed ideal of \( X \) generated by \( \nu \). Clearly \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \). Let \( x,y \in X \). Then there exist \( i \) and \( j \) in \( \{1,2,\ldots,n\} \) such that \( x \ast y \in D_i \) and \( y \in D_j \). Without loss of generality, we may assume that \( i \) and \( j \) are the smallest integers such that \( i \geq j \), \( x \ast y \in D_i \), and \( y \in D_j \). Since \( D_i \) is a closed ideal of \( X \), it follows from \( D_j \subseteq D_i \) that \( x \in D_i \). Hence \( \mu(x) \geq \alpha_i = \min\{\mu(x \ast y),\mu(y)\} \), and so \( \mu \) is a fuzzy closed ideal of \( X \). If \( \nu(x) = \alpha_j \) for every \( x \in X \), then \( x \in C_j \) and thus \( x \in D_j \). But we have \( \mu(x) \geq \alpha_j = \nu(x) \). Therefore \( \mu \) contains \( \nu \). Let \( \delta : X \to [0,1] \) be a fuzzy closed ideal of \( X \) containing \( \nu \). Then \( U(\nu;\alpha_j) \subseteq U(\delta;\alpha_j) \) for every \( j \). Hence \( U(\delta;\alpha_j) \), being a closed ideal, contains the closed ideal generated by \( U(\nu;\alpha_j) = \cup_{i=1}^j C_i \). Consequently, \( D_j \subseteq U(\delta;\alpha_j) \). It follows that \( \mu \) is contained in \( \delta \) and that \( \mu \) is generated by \( \nu \). Finally, note that \( |\text{Im}(\mu)| = n = |\text{Im}(\nu)| \). This completes the proof.

\( \square \)

Theorem 3.10. Let \( D_1 \supseteq D_2 \supseteq \cdots \) be a descending chain of closed ideals of \( X \) which terminates at finite step. For a fuzzy closed ideal \( \mu \) of \( X \), if a sequence of elements of \( \text{Im}(\mu) \) is strictly increasing, then \( \mu \) is finite valued.

Proof. Suppose that \( \mu \) is infinite valued. Let \( \{\alpha_n\} \) be a strictly increasing sequence of elements of \( \text{Im}(\mu) \). Then \( 0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1 \). Note that \( U(\mu;\alpha_t) \) is a closed ideal of \( X \) for \( t = 1,2,3,\ldots \). Let \( x \in U(\mu;\alpha_t) \) for \( t = 2,3,\ldots \). Then \( \mu(x) \geq \alpha_t > \alpha_{t-1} \), which implies that \( x \in U(\mu;\alpha_{t-1}) \). Hence \( U(\mu;\alpha_t) \subseteq U(\mu;\alpha_{t-1}) \) for \( t = 2,3,\ldots \). Since \( \alpha_{t-1} \in \text{Im}(\mu) \), there exists \( x_{t-1} \in X \) such that \( \mu(x_{t-1}) = \alpha_{t-1} \). It follows that \( x_{t-1} \in U(\mu;\alpha_{t-1}) \), but \( x_{t-1} \notin U(\mu;\alpha_t) \). Thus \( U(\mu;\alpha_t) \nsubseteq U(\mu;\alpha_{t-1}) \), and so we obtain a strictly descending chain \( U(\mu;\alpha_t) \supseteq U(\mu;\alpha_2) \supseteq \cdots \) of closed ideals of \( X \) which is not terminating. This is impossible and the proof is complete.

Now we consider the converse of Theorem 3.10.

Theorem 3.11. Let \( \mu \) be a finite-valued fuzzy closed ideal of \( X \). Then every descending chain of closed ideals of \( X \) terminates at finite step.
**Proof.** Suppose there exists a strictly descending chain $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \cdots$ of closed ideals of $X$ which does not terminate at finite step. Define a fuzzy set $\mu$ in $X$ by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in D_n \setminus D_{n+1}, \ n = 0, 1, 2, \ldots, \\ 1 & \text{if } x \in \cap_{n=0}^{\infty} D_n, \end{cases}$$

where $D_0$ stands for $X$. Clearly, $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x \ast y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $y \in D_n$, and so $x \in D_n$ because $D_n$ is a closed ideal of $X$. Hence

$$\mu(x) \geq \frac{n}{n+1} = \min \{ \mu(x \ast y), \mu(y) \}.$$  \hfill (3.19)

If $x \ast y \in \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then $x \in \cap_{n=0}^{\infty} D_n$. Thus $\mu(x) = 1 = \min \{ \mu(x \ast y), \mu(y) \}$. If $x \ast y \notin \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then there exists a positive integer $k$ such that $x \ast y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$\mu(x) \geq \frac{k}{k+1} = \min \{ \mu(x \ast y), \mu(y) \}. \hfill (3.20)$$

Finally suppose that $x \ast y \in \cap_{n=0}^{\infty} D_n$ and $y \notin \cap_{n=0}^{\infty} D_n$. Then $y \in D_r \setminus D_{r+1}$ for some positive integer $r$. It follows that $x \in D_r$, and hence

$$\mu(x) \geq \frac{r}{r+1} = \min \{ \mu(x \ast y), \mu(y) \}. \hfill (3.21)$$

Consequently, we conclude that $\mu$ is a fuzzy closed ideal of $X$ and $\mu$ has an infinite number of different values. This is a contradiction, and the proof is complete. \hfill $\Box$

**Theorem 3.12.** The following are equivalent:

(i) Every ascending chain of closed ideals of $X$ terminates at finite step.

(ii) The set of values of any fuzzy closed ideal of $X$ is a well-ordered subset of $[0, 1]$.

**Proof.** (i)$\Rightarrow$(ii). Let $\mu$ be a fuzzy closed ideal of $X$. Suppose that the set of values of $\mu$ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence \{\alpha_n\} such that $\mu(x_n) = \alpha_n$. It follows that

$$U(\mu; \alpha_1) \subseteq U(\mu; \alpha_2) \subseteq U(\mu; \alpha_3) \subseteq \cdots$$ \hfill (3.22)

is a strictly ascending chain of closed ideals of $X$. This is impossible.

(ii)$\Rightarrow$(i). Assume that there exists a strictly ascending chain

$$D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$$ \hfill (3.23)

of closed ideals of $X$. Note that $D := \cup_{n \in \mathbb{N}} D_n$ is a closed ideal of $X$. Define a fuzzy set $\mu$ in $X$ by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin D_n, \\ 1 & \text{if } x \in D_n, \end{cases}$$

where $k = \min \{ n \in \mathbb{N} \mid x \in D_n \}$.
We claim that $\mu$ is a fuzzy closed ideal of $X$. Let $x \in X$. If $x \not\in D_n$, then obviously $\mu(0 \ast x) \geq 0 = \mu(x)$. If $x \in D_n \setminus D_{n-1}$ for $n = 2, 3, \ldots$, then $0 \ast x \in D_n$. Hence $\mu(0 \ast x) \geq 1/n = \mu(x)$. Let $x, y \in X$. If $x \ast y \in D_n \setminus D_{n-1}$ and $y \in D_n \setminus D_{n-1}$ for $n = 2, 3, \ldots$, then $x \in D_n$. It follows that

$$\mu(x) \geq \frac{1}{n} = \min\{\mu(x \ast y), \mu(y)\}. \quad (3.26)$$

Suppose that $x \ast y \in D_n$ and $y \in D_n \setminus D_m$ for all $m < n$. Then $x \in D_n$, and so $\mu(x) \geq 1/n \geq 1/m + 1 \geq \mu(y)$. Hence $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$. Similarly for the case $x \ast y \in D_n \setminus D_m$ and $y \in D_n$, we get $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$. Therefore $\mu$ is a fuzzy closed ideal of $X$. Since the chain (3.24) is not terminating, $\mu$ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well ordered. This completes the proof. \hfill \Box

4. $T$-fuzzy subalgebras and $T$-fuzzy closed ideals

**Definition 4.1.** A fuzzy set $\mu$ in $X$ is said to satisfy *imaginable property* if $\text{Im}(\mu) \subseteq \Delta_T$.

**Definition 4.2.** A fuzzy set $\mu$ in $X$ is called a *fuzzy subalgebra* of $X$ with respect to a $t$-norm $T$ (briefly, $T$-fuzzy subalgebra of $X$) if $\mu(x \ast y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$. A $T$-fuzzy subalgebra of $X$ is said to be *imaginable* if it satisfies the imaginable property.

**Example 4.3.** Let $T_m$ be a $t$-norm defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$ and let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 
0.9 & \text{if } x \in \{0, d\}, \\
0.09 & \text{otherwise}. 
\end{cases} \quad (4.1)$$

Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$, which is not imaginable.

(2) Let $\nu$ be a fuzzy set in $X$ defined by

$$\nu(x) = \begin{cases} 
1 & \text{if } x \in \{0, d\}, \\
0 & \text{otherwise}. 
\end{cases} \quad (4.2)$$

Then $\nu$ is an imaginable $T_m$-fuzzy subalgebra of $X$. 

**Proposition 4.4.** Let $A$ be a subalgebra of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$
\mu(x) := \begin{cases} 
\alpha_1 & \text{if } x \in A, \\
\alpha_2 & \text{otherwise}, 
\end{cases}
$$

for all $x \in X$, where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$. In particular, if $\alpha_1 = 1$ and $\alpha_2 = 0$ then $\mu$ is an imaginable $T_m$-fuzzy subalgebra of $X$, where $T_m$ is the $t$-norm in Example 4.3.

**Proof.** Let $x, y \in X$. If $x \in A$ and $y \in A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_1) = \max (2\alpha_1 - 1, 0) = \begin{cases} 
2\alpha_1 - 1 & \text{if } \alpha_1 \geq \frac{1}{2} \\
0 & \text{if } \alpha_1 < \frac{1}{2} 
\end{cases}
$$

(4.4)

If $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$) then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_2) = \max (\alpha_1 + \alpha_2 - 1, 0) = \begin{cases} 
\alpha_1 + \alpha_2 - 1 & \text{if } \alpha_1 + \alpha_2 \geq 1 \\
0 & \text{otherwise} 
\end{cases}
$$

(4.5)

$$
\leq \alpha_2 \leq \mu(x \ast y).
$$

If $x, y \notin A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(2\alpha_2 - 1, 0) = \begin{cases} 
2\alpha_2 - 1 & \text{if } \alpha_2 \geq \frac{1}{2} \\
0 & \text{if } \alpha_2 < \frac{1}{2} 
\end{cases}
$$

(4.6)

$$
\leq \alpha_2 \leq \mu(x \ast y).
$$

Hence $\mu$ is a $T_m$-fuzzy subalgebra of $X$. Assume that $\alpha_1 = 1$ and $\alpha_2 = 0$. Then

$$
T_m(\alpha_1, \alpha_1) = \max (\alpha_1 + \alpha_1 - 1, 0) = 1 = \alpha_1,
$$

$$
T_m(\alpha_2, \alpha_2) = \max (\alpha_2 + \alpha_2 - 1, 0) = 0 = \alpha_2.
$$

(4.7)

Thus $\alpha_1, \alpha_2 \in \Delta_{T_m}$, that is, $\text{Im}(\mu) \subseteq \Delta_{T_m}$ and so $\mu$ is imaginable. This completes the proof.

**Proposition 4.5.** If $\mu$ is an imaginable $T$-fuzzy subalgebra of $X$, then $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. 

Proof. For any \( x \in X \) we have
\[
\mu(0 \ast x) \geq T(\mu(0), \mu(x))
\]
\[
= T(\mu(x \ast x), \mu(x)) \quad \text{[by (H1)]}
\]
\[
\geq T(T(\mu(x), \mu(x)), \mu(x)) \quad \text{[by (T2) and (T3)]}
\]
\[
= \mu(x), \quad \text{[since } \mu \text{ satisfies the imaginable property].}
\]
(4.8)
This completes the proof. \( \square \)

Theorem 4.6. Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \) and let \( \alpha \in [0, 1] \) be such that \( T(\alpha, \alpha) = \alpha \). Then \( U(\mu; \alpha) \) is either empty or a subalgebra of \( X \), and moreover \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

Proof. Let \( x, y \in U(\mu; \alpha) \). Then
\[
\mu(x \ast y) \geq \mu(x) \ast \mu(y) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha,
\]
which implies that \( x \ast y \in U(\mu; \alpha) \). Hence \( U(\mu; \alpha) \) is a subalgebra of \( X \). Since \( x \ast x = 0 \) for all \( x \in X \), we have \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) = \mu(x) \) for all \( x \in X \). \( \square \)

Since \( T(1, 1) = 1 \), we have the following corollary.

Corollary 4.7. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( U(\mu; 1) \) is either empty or a subalgebra of \( X \).

Theorem 4.8. Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If there is a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), then \( \mu(0) = 1 \).

Proof. Let \( x \in X \). Then \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) \). Therefore \( \mu(0) \geq T(\mu(x_n), \mu(x_n)) \) for each \( n \in \mathbb{N} \). Since \( 1 \geq \mu(0) \geq \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), it follows that \( \mu(0) = 1 \), this completes the proof. \( \square \)

Let \( f : X \to Y \) be a mapping of BCH-algebras. For a fuzzy set \( \mu \) in \( Y \), the inverse image of \( \mu \) under \( f \), denoted by \( f^{-1}(\mu) \), is defined by \( f^{-1}(\mu)(x) = \mu(f(x)) \) for all \( x \in X \).

Theorem 4.9. Let \( f : X \to Y \) be a homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( Y \), then \( f^{-1}(\mu) \) is a \( T \)-fuzzy subalgebra of \( X \).

Proof. For any \( x, y \in X \), we have
\[
f^{-1}(\mu)(x \ast y) = \mu(f(x \ast y)) = \mu(f(x) \ast f(y))
\]
\[
\geq T(\mu(f(x)), \mu(f(y)))
\]
\[
= T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)).
\]
(4.10)
This completes the proof. \( \square \)

If \( \mu \) is a fuzzy set in \( X \) and \( f \) is a mapping defined on \( X \). The fuzzy set \( f(\mu) \) in \( f(X) \) defined by \( f(\mu)(y) = \sup \{ \mu(x) \mid x \in f^{-1}(y) \} \) for all \( y \in f(X) \) is called the image of \( \mu \) under \( f \). A fuzzy set \( \mu \) in \( X \) is said to have \( sup \) property if, for every subset \( T \subseteq X \), there exists \( t_0 \in T \) such that \( \mu(t_0) = \sup \{ \mu(t) \mid t \in T \} \).
**Theorem 4.10.** An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.

**Proof.** Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras and let \( \mu \) be a fuzzy subalgebra of \( X \) with sup property. Given \( u, v \in Y \), let \( x_0 \in f^{-1}(u) \) and \( y_0 \in f^{-1}(v) \) be such that

\[
\mu(x_0) = \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \quad \mu(y_0) = \sup \{ \mu(t) \mid t \in f^{-1}(v) \},
\]

respectively. Then

\[
f(\mu)(u \ast v) = \sup \{ \mu(z) \mid z \in f^{-1}(u \ast v) \}
\geq \min \{ \mu(x_0), \mu(y_0) \}
= \min \{ \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \sup \{ \mu(t) \mid t \in f^{-1}(v) \} \}
= \min \{ f(\mu)(u), f(\mu)(v) \}.
\]

Hence \( f(\mu) \) is a fuzzy subalgebra of \( Y \).

**Theorem 4.10** can be strengthened in the following way. To do this we need the following definition.

**Definition 4.11.** A \( t \)-norm \( T \) on \([0, 1]\) is called a continuous \( t \)-norm if \( T \) is a continuous function from \([0, 1] \times [0, 1]\) to \([0, 1]\) with respect to the usual topology.

Note that the function “\( \min \)” is a continuous \( t \)-norm.

**Theorem 4.12.** Let \( T \) be a continuous \( t \)-norm and let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

**Proof.** Let \( A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2), \) and \( A_{12} = f^{-1}(y_1 \ast y_2), \) where \( y_1, y_2 \in Y \). Consider the set

\[
A_1 \ast A_2 := \{ x \in X \mid x = a_1 \ast a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2 \}.
\]

If \( x \in A_1 \ast A_2 \), then \( x = x_1 \ast x_2 \) for some \( x_1 \in A_1 \) and \( x_2 \in A_2 \) and so

\[
f(x) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2,
\]

that is, \( x \in f^{-1}(y_1 \ast y_2) = A_{12} \). Thus \( A_1 \ast A_2 \subseteq A_{12} \). It follows that

\[
f(\mu)(y_1 \ast y_2) = \sup \{ \mu(x) \mid x \in f^{-1}(y_1 \ast y_2) \}
\geq \sup \{ \mu(x) \mid x \in A_1 \ast A_2 \}
\geq \sup \{ \mu(x_1 \ast x_2) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}.
\]

Since \( T \) is continuous, for every \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that if \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - x_1^\# \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - x_2^\# \leq \delta \) then

\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(x_1^\#, x_2^\#) \leq \varepsilon.
\]
Choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \) such that \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - \mu(a_1) \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - \mu(a_2) \leq \delta \). Then
\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(\mu(a_1), \mu(a_2)) \leq \varepsilon. \tag{4.17}
\]
Consequently
\[
f(\mu)(\gamma_1 * \gamma_2) \geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \})
= T(f(\mu)(y_1), f(\mu)(y_2)),
\tag{4.18}
\]
which shows that \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

**Lemma 4.13** (see [1]). For all \( \alpha, \beta, y, \delta \in [0,1] \),
\[
T(T(\alpha, \beta), T(y, \delta)) = T(T(\alpha, y), T(\beta, \delta)). \tag{4.19}
\]

**Theorem 4.14.** Let \( X = X_1 \times X_2 \) be the direct product BCH-algebra of BCH-algebras \( X_1 \) and \( X_2 \). If \( \mu_1 \) (resp., \( \mu_2 \)) is a \( T \)-fuzzy subalgebra of \( X_1 \) (resp., \( X_2 \)), then \( \mu = \mu_1 \times \mu_2 \) is a \( T \)-fuzzy subalgebra of \( X \) defined by
\[
\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)), \tag{4.20}
\]
for all \( (x_1, x_2) \in X_1 \times X_2 \).

**Proof.** Let \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) be any elements of \( X = X_1 \times X_2 \). Then
\[
\begin{align*}
\mu(x * y) &= \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1, y_1, x_2, y_2) \\
&= T(\mu_1(x_1, y_1), \mu_2(x_2, y_2)) \\
&\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) \\
&= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) \\
&= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) \\
&= T(\mu(x_1, x_2), \mu(x_2, y_2)) \\
&= T(\mu(x_1, x_2), \mu(x_2, y_2)) \\
&= T(\mu(x), \mu(y)).
\end{align*}
\]
Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

We will generalize the idea to the product of \( n \) \( T \)-fuzzy subalgebras. We first need to generalize the domain of \( T \) to \( \prod_{i=1}^{n} [0,1] \) as follows:

**Definition 4.15** (see [1]). The function \( T_n : \prod_{i=1}^{n} [0,1] \to [0,1] \) is defined by
\[
T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_1, T_{n-1}(\alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)), \tag{4.22}
\]
for all \( 1 \leq i \leq n \), where \( n \geq 2 \), \( T_2 = T \), and \( T_1 = \text{id} \) (identity).

**Lemma 4.16** (see [1]). For every \( \alpha_i, \beta_i \in [0,1] \) where \( 1 \leq i \leq n \) and \( n \geq 2 \),
\[
T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta_1, \beta_2, \ldots, \beta_n)). \tag{4.23}
\]
**Theorem 4.17.** Let \( \{X_i\}_{i=1}^n \) be the finite collection of BCH-algebras and \( X = \prod_{i=1}^n X_i \) the direct product BCH-algebra of \( \{X_i\} \). Let \( \mu_i \) be a \( T \)-fuzzy subalgebra of \( X_i \), where \( 1 \leq i \leq n \). Then \( \mu = \prod_{i=1}^n \mu_i \) defined by

\[
\mu(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^n \mu_i \right)(x_1, x_2, \ldots, x_n)
\]

is a \( T \)-fuzzy subalgebra of the BCH-algebra \( X \).

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be any elements of \( X = \prod_{i=1}^n X_i \). Then

\[
\mu(x \ast y) = \mu(x_1 \ast y_1, x_2 \ast y_2, \ldots, x_n \ast y_n)
= T_n(\mu_1(x_1 \ast y_1), \mu_2(x_2 \ast y_2), \ldots, \mu_n(x_n \ast y_n))
\geq T_n(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2)), \ldots, T(\mu_n(x_n), \mu_n(y_n)))
= T(T_\mu(x_1), T_\mu(x_2), \ldots, T_\mu(x_n), T(\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n)))
= T(T_\mu(x), T_\mu(y)).
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

**Definition 4.18.** Let \( \mu \) and \( \nu \) be fuzzy sets in \( X \). Then the \( T \)-product of \( \mu \) and \( \nu \), written \([\mu \ast \nu]_T\), is defined by \([\mu \ast \nu]_T(x) = T(\mu(x), \nu(x))\) for all \( x \in X \).

**Theorem 4.19.** Let \( \mu \) and \( \nu \) be \( T \)-fuzzy subalgebras of \( X \). If \( T^* \) is a \( t \)-norm which dominates \( T \), that is,

\[
T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T^*(T(\alpha, \gamma), T^*(\beta, \delta)),
\]

for all \( \alpha, \beta, \gamma, \delta \in [0, 1] \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \([\mu \ast \nu]_{T^*}\), is a \( T \)-fuzzy subalgebra of \( X \).

**Proof.** For any \( x, y \in X \) we have

\[
[\mu \ast \nu]_{T^*}(x \ast y) = T^*(\mu(x \ast y), \nu(x \ast y))
\geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))
\geq T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y)))
= T([\mu \ast \nu]_{T^*}(x), [\mu \ast \nu]_{T^*}(y)).
\]

Hence \([\mu \ast \nu]_{T^*}\) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

Let \( f : X \rightarrow Y \) be an onto homomorphism of BCH-algebras. Let \( T \) and \( T^* \) be \( t \)-norms such that \( T^* \) dominates \( T \). If \( \mu \) and \( \nu \) are \( T \)-fuzzy subalgebras of \( Y \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \([\mu \ast \nu]_{T^*}\), is a \( T \)-fuzzy subalgebra of \( Y \). Since every onto homomorphic inverse image of a \( T \)-fuzzy subalgebra is a \( T \)-fuzzy subalgebra, the
inverse images \( f^{-1}(\mu), f^{-1}(\nu), \) and \( f^{-1}((\mu \cdot \nu)^* \cdot T) \) are \( T \)-fuzzy subalgebras of \( X \). The next theorem provides that the relation between \( f^{-1}((\mu \cdot \nu)^* \cdot T) \) and the \( T^* \)-product 
\[ f^{-1}(\mu) \cdot f^{-1}(\nu) \] of \( f^{-1}(\mu) \) and \( f^{-1}(\nu) \).

**Theorem 4.20.** Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. Let \( T^* \) be a \( t \)-norm such that \( T^* \) dominates \( T \). Let \( \mu \) and \( \nu \) be \( T^* \)-fuzzy subalgebras of \( Y \). If 
\[ [\mu \cdot \nu]^{T*} \] is the \( T^* \)-product of \( \mu \) and \( \nu \) and 
\[ [f^{-1}(\mu) \cdot f^{-1}(\nu)]^{T*} \] is the \( T^* \)-product of \( f^{-1}(\mu) \) and \( f^{-1}(\nu) \), then
\[ f^{-1}([\mu \cdot \nu]^{T*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]^{T*}. \tag{4.28} \]

**Proof.** For any \( x \in X \) we get
\[ f^{-1}([\mu \cdot \nu]^{T*})(x) = [\mu \cdot \nu]^{T*}(f(x)) = T^*(\mu(f(x)), \nu(f(x))) = T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]^{T*}(x), \tag{4.29} \]
This completes the proof. \( \square \)

**Definition 4.21.** A fuzzy set \( \mu \) in \( X \) is called a fuzzy closed ideal of \( X \) under a \( t \)-norm \( T \) (briefly, \( T \)-fuzzy closed ideal of \( X \)) if
(F1) \( \mu(0 \cdot x) \geq \mu(x) \) for all \( x \in X \),
(F2) \( \mu(x) \geq T(\mu(x \cdot y), \mu(y)) \) for all \( x, y \in X \).

A \( T \)-fuzzy closed ideal of \( X \) is said to be imaginable if it satisfies the imaginable property.

**Example 4.22.** Let \( T_m \) be a \( t \)-norm in Example 4.3. Consider a BCH-algebra \( X = \{0, a, b, c\} \) with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set \( \mu : X \to [0, 1] \) by \( \mu(0) = \mu(c) = 0.8 \) and \( \mu(a) = \mu(b) = 0.3 \).
Then \( \mu \) is a \( T_m \)-fuzzy closed ideal of \( X \) which is not imaginable.
(2) Let \( \nu \) be a fuzzy set in \( X \) defined by
\[ \nu(x) = \begin{cases} 
1 & \text{if } x \in \{0, c\}, \\
0 & \text{otherwise.} 
\end{cases} \tag{4.30} \]
Then \( \nu \) is an imaginable \( T_m \)-fuzzy closed ideal of \( X \).

**Theorem 4.23.** Every imaginable \( T \)-fuzzy subalgebra satisfying (F3) is an imaginable \( T \)-fuzzy closed ideal.

**Proof.** Using Proposition 4.5, it is straightforward. \( \square \)
**Proposition 4.24.** If \( \mu \) is an imaginable \( T \)-fuzzy closed ideal of \( X \), then \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

**Proof.** Using (F1), (F3), and (T2), we have

\[
\mu(0) \geq T(\mu(0 \ast x), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x)
\]

for all \( x \in X \), completing the proof.

**Theorem 4.25.** Every \( T \)-fuzzy closed ideal is a \( T \)-fuzzy subalgebra.

**Proof.** Let \( \mu \) be a \( T \)-fuzzy closed ideal of \( X \) and let \( x, y \in X \). Then

\[
\mu(x \ast y) \geq T(\mu((x \ast y) \ast x), \mu(x)) \quad \text{[by (F3)]}
\]

\[
= T(\mu((x \ast x) \ast y), \mu(x)) \quad \text{[by (H3)]}
\]

\[
= T(\mu(0 \ast y), \mu(x)) \quad \text{[by (H1)]}
\]

\[
\geq T(\mu(x), \mu(y)) \quad \text{[by (F1), (T2), and (T3)].}
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

The converse of Theorem 4.25 may not be true. For example, the \( T_m \)-fuzzy subalgebra \( \mu \) in Example 4.3(1) is not a \( T_m \)-fuzzy closed ideal of \( X \) since

\[
\mu(a) = 0.09 < 0.9 = T_m(\mu(a \ast d), \mu(d)).
\]

We give a condition for a \( T \)-fuzzy subalgebra to be a \( T \)-fuzzy closed ideal.

**Theorem 4.26.** Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If \( \mu \) satisfies the imaginable property and the inequality

\[
\mu(x \ast y) \leq \mu(y \ast x) \quad \forall x, y \in X,
\]

then \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

**Proof.** Let \( \mu \) be an imaginable \( T \)-fuzzy subalgebra of \( X \) which satisfies the inequality

\[
\mu(x \ast y) \leq \mu(y \ast x) \quad \forall x, y \in X.
\]

It follows from *Proposition 4.5* that \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Then

\[
\mu(x) = \mu(x \ast 0) \geq \mu(0 \ast x) = \mu((y \ast y) \ast x)
\]

\[
= \mu((y \ast x) \ast y) \geq T(\mu(y \ast x), \mu(y)) \geq T(\mu(x \ast y), \mu(y)).
\]

Hence \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

**Proposition 4.27.** Let \( T_m \) be a \( t \)-norm in Example 4.3. Let \( D \) be a closed ideal of \( X \) and let \( \mu \) be a fuzzy set in \( X \) defined by

\[
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D, \\
\alpha_2 & \text{otherwise},
\end{cases}
\]

for all \( x \in X \).
(i) If $\alpha_1 = 1$ and $\alpha_2 = 0$, then $\mu$ is an imaginable $T_m$-fuzzy closed ideal of $X$.

(ii) If $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 > \alpha_2$, then $\mu$ is a $T_m$-fuzzy closed ideal of $X$ which is not imaginable.

**Proof.** (i) If $x \in D$, then $0 \ast x \in D$ and so $\mu(0 \ast x) = 1 = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0 \ast x)$. Now obviously if $x \in D$, then

$$
\mu(x) = 1 \geq T_m(\mu(x \ast y), \mu(y)),
$$

(4.38)

for all $y \in X$. Assume that $x \notin D$. Then $x \ast y \notin D$ or $y \notin D$, that is, $\mu(x \ast y) = 0$ or $\mu(y) = 0$. It follows that

$$
T_m(\mu(x \ast y), \mu(y)) = 0 = \mu(x).
$$

(4.39)

Hence $\mu(x) \geq T_m(\mu(x \ast y), \mu(y))$ for all $x, y \in X$. Clearly $\text{Im}(\mu) \subseteq T_m$.

(ii) Similar to (i), we know that $\mu$ is a $T_m$-fuzzy closed ideal of $X$. Taking $\alpha_1 = 0.7$, then

$$
T_m(\alpha_1, \alpha_1) = T_m(0.7, 0.7) = \max(0.7 + 0.7 - 1, 0) = 0.4 \neq \alpha_1.
$$

(4.40)

Hence $\alpha_1 \notin T_m$, that is, $\text{Im}(\mu) \not\subseteq T_m$, and so $\mu$ is not imaginable. \hfill \Box

**Proposition 4.28.** Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$. If $\mu$ satisfies the inequality $\mu(x) \geq \mu(0 \ast x)$ for all $x \in X$, then it satisfies the equality $\mu(x \ast y) = \mu(y \ast x)$ for all $x, y \in X$.

**Proof.** Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$ satisfying the inequality $\mu(x) \geq \mu(0 \ast x)$ for all $x \in X$. For every $x, y \in X$, we have

$$
\mu(y \ast x) \geq \mu(0 \ast (y \ast x)) \quad \text{[by assumption]}
$$

\[\begin{align*}
&\geq T(\mu(((0 \ast (y \ast x)) \ast (x \ast y)), \mu(x \ast y)) \quad \text{[by (F3)]} \\
&= T(\mu(((0 \ast y) \ast (0 \ast x)) \ast (x \ast y)), \mu(x \ast y)) \quad \text{[by (P3)]} \\
&= T(\mu(((0 \ast y) \ast (x \ast y)) \ast (0 \ast x)), \mu(x \ast y)) \quad \text{[by (H3)]} \\
&= T(\mu(((0 \ast (x \ast y)) \ast y) \ast (0 \ast x)), \mu(x \ast y)) \quad \text{[by (H3)]} \\
&= T(\mu(((0 \ast x) \ast (0 \ast y)) \ast y), \mu(x \ast y)) \quad \text{[by (P3)]} \\
&= T(\mu(((0 \ast x) \ast (0 \ast y)) \ast (0 \ast x)) \ast y), \mu(x \ast y)) \quad \text{[by (H3)]} \\
&= T(\mu((0 \ast (0 \ast y)) \ast y), \mu(x \ast y)) \quad \text{[by (H1)]} \\
&= T(\mu(0), \mu(x \ast y)) \quad \text{[by (H3) and (H1)]} \\
&\geq T(\mu(x \ast y), \mu(x \ast y)), \mu(y \ast y)) \quad \text{[by (H1)]} \\
&\geq T(T(\mu(x \ast y), \mu(x \ast y)), \mu(x \ast y)) \quad \text{[by Proposition 4.24 and (T2)]} \\
&= \mu(x \ast y) \quad \text{[since $\mu$ is imaginable].}
\end{align*}\]

(4.41)

Similarly we have $\mu(x \ast y) \geq \mu(y \ast x)$ for all $x, y \in X$, completing the proof. \hfill \Box
Theorem 4.29. Every imaginable $T$-fuzzy closed ideal is a fuzzy closed ideal.

Proof. Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$. Then

$$\mu(x) \geq T(\mu(x \ast y), \mu(y)) \quad \forall x, y \in X.$$  \hspace{1cm} (4.42)

Since $\mu$ is imaginable, we have

$$\min(\mu(x \ast y), \mu(y)) = T(\min(\mu(x \ast y), \mu(y)), \min(\mu(x \ast y), \mu(y)))$$

$$\leq T(\mu(x \ast y), \mu(y))$$

$$\leq \min(\mu(x \ast y), \mu(y)).$$  \hspace{1cm} (4.43)

It follows that $\mu(x) \geq T(\mu(x \ast y), \mu(y)) = \min(\mu(x \ast y), \mu(y))$ so that $\mu$ is a fuzzy closed ideal of $X$.

Combining Theorems 3.3, 4.29, we have the following corollary.

Corollary 4.30. If $\mu$ is an imaginable $T$-fuzzy closed ideal of $X$, then the nonempty level set of $\mu$ is a closed ideal of $X$.

Noticing that the fuzzy set $\mu$ in Example 4.22(1) is a fuzzy closed ideal of $X$, we know from Example 4.22(1) that there exists a $t$-norm such that the converse of Theorem 4.29 may not be true.

Proposition 4.31. Every imaginable $T$-fuzzy closed ideal is order reversing.

Proof. Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$ and let $x, y \in X$ be such that $x \leq y$. Using (P4), (T2), Theorem 4.29, Proposition 4.24, and the definition of a fuzzy closed ideal, we get

$$\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \geq T(\mu(x \ast y), \mu(y))$$

$$= T(\mu(x), \mu(y)) \geq T(\mu(y), \mu(y)) = \mu(y).$$  \hspace{1cm} (4.44)

This completes the proof.

Proposition 4.32. Let $\mu$ be a $T$-fuzzy closed ideal of $X$, where $T$ is a diagonal $t$-norm on $[0,1]$, that is, $T(\alpha, \alpha) = \alpha$ for all $\alpha \in [0,1]$. If $(x \ast a) \ast b = 0$ for all $a, b, x \in X$, then $\mu(x) \geq T(\mu(a), \mu(b))$.

Proof. Let $a, b, x \in X$ be such that $(x \ast a) \ast b = 0$. Then

$$\mu(x) \geq T(\mu(x \ast a), \mu(a))$$

$$\geq T(T(\mu((x \ast a) \ast b), \mu(b)), \mu(a))$$

$$= T(\mu(0), \mu(b)), \mu(a))$$

$$\geq T(\mu(b), \mu(b)), \mu(a))$$

$$= T(\mu(a), \mu(b)),$$  \hspace{1cm} (4.45)

completing the proof.
**Corollary 4.33.** Let \( \mu \) be a \( T \)-fuzzy closed ideal of \( X \), where \( T \) is a diagonal \( t \)-norm on \([0,1]\). If \( (\cdots ((x \ast a_1) \ast a_2) \ast \cdots) \ast a_n = 0 \) for all \( x, a_1, a_2, \ldots, a_n \in X \), then

\[
\mu(x) \geq T_n(\mu(a_1), \mu(a_2), \ldots, \mu(a_n)).
\]

**Proof.** Using induction on \( n \), the proof is straightforward. \( \square \)

**Theorem 4.34.** There exists a \( t \)-norm \( T \) such that every closed ideal of \( X \) can be realized as a level closed ideal of a \( T \)-fuzzy closed ideal of \( X \).

**Proof.** Let \( D \) be a closed ideal of \( X \) and let \( \mu \) be a fuzzy set in \( X \) defined by

\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in D, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \alpha \in (0,1) \) is fixed. It is clear that \( U(\mu; \alpha) = D \). We will prove that \( \mu \) is a \( T_m \)-fuzzy closed ideal of \( X \), where \( T_m \) is a \( t \)-norm in Example 4.3. If \( x \in D \), then \( 0 \ast x \in D \) and so \( \mu(0 \ast x) = \alpha = \mu(x) \). If \( x \notin D \), then clearly \( \mu(x) = 0 \leq \mu(0 \ast x) \). Let \( x, y \in X \). If \( x \in D \), then \( \mu(x) = \alpha \geq T_m(\mu(x \ast y), \mu(y)) \). If \( x \notin D \), then \( x \ast y \notin D \) or \( y \notin D \). It follows that \( \mu(x) = 0 = T_m(\mu(x \ast y), \mu(y)) \). This completes the proof. \( \square \)

For a family \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \) of fuzzy sets in \( X \), define the join \( \bigvee_{\alpha \in \Lambda} \mu_\alpha \) and the meet \( \bigwedge_{\alpha \in \Lambda} \mu_\alpha \) as follows:

\[
(\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \quad (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \},
\]

for all \( x \in X \), where \( \Lambda \) is any index set.

**Theorem 4.35.** The family of \( T \)-fuzzy closed ideals in \( X \) is a completely distributive lattice with respect to meet "\( \wedge \)" and the join "\( \vee \)."

**Proof.** Since \([0,1]\) is a completely distributive lattice with respect to the usual ordering in \([0,1]\), it is sufficient to show that \( \bigvee_{\alpha \in \Lambda} \mu_\alpha \) and \( \bigwedge_{\alpha \in \Lambda} \mu_\alpha \) are \( T \)-fuzzy closed ideals of \( X \) for a family of \( T \)-fuzzy closed ideals \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \). For any \( x \in X \), we have

\[
(\bigvee_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \sup \{ \mu_\alpha(0 \ast x) \mid \alpha \in \Lambda \}
\geq \sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}
= (\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x),
\]

\[
(\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \inf \{ \mu_\alpha(0 \ast x) \mid \alpha \in \Lambda \}
\geq \inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}
= (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x).
\]

Let \( x, y \in X \). Then

\[
(\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}
\geq \sup \{ T(\mu_\alpha(x \ast y), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\geq T(\sup \{ \mu_\alpha(x \ast y) \mid \alpha \in \Lambda \}, \sup \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
= T((\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x \ast y), (\bigvee_{\alpha \in \Lambda} \mu_\alpha)(y)),
\]

\[
(\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}
\leq \inf \{ T(\mu_\alpha(x \ast y), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\leq T(\inf \{ \mu_\alpha(x \ast y) \mid \alpha \in \Lambda \}, \inf \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
= T((\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x \ast y), (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(y)),
\]

\[
(\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x \ast y) = \sup \{ \mu_\alpha(x \ast y) \mid \alpha \in \Lambda \}
\geq \sup \{ T(\mu_\alpha(x), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\geq T(\sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \sup \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
= T((\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x), (\bigvee_{\alpha \in \Lambda} \mu_\alpha)(y)),
\]

\[
(\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x \ast y) = \inf \{ \mu_\alpha(x \ast y) \mid \alpha \in \Lambda \}
\leq \inf \{ T(\mu_\alpha(x), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\leq T(\inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \inf \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
= T((\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x), (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(y)).
\]
\[
(\land_{\alpha \in \Lambda} \mu_{\alpha})(x) = \inf \{ \mu_{\alpha}(x) \mid \alpha \in \Lambda \}
\geq \inf \{ T(\mu_{\alpha}(x \ast y), \mu_{\alpha}(y)) \mid \alpha \in \Lambda \}
\geq T(\inf \{ \mu_{\alpha}(x \ast y) \mid \alpha \in \Lambda \}, \inf \{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \})
= T((\land_{\alpha \in \Lambda} \mu_{\alpha})(x \ast y), (\land_{\alpha \in \Lambda} \mu_{\alpha})(y)).
\]

(4.50)

Hence \( \lor_{\alpha \in \Lambda} \mu_{\alpha} \) and \( \land_{\alpha \in \Lambda} \mu_{\alpha} \) are \( T \)-fuzzy closed ideals of \( X \), completing the proof. \( \square \)

5. Conclusions and future works. We inquired into further properties on fuzzy closed ideals in BCH-algebras, and using a \( t \)-norm \( T \), we introduced the notion of (imaginable) \( T \)-fuzzy subalgebras and (imaginable) \( T \)-fuzzy closed ideals, and obtained some related results. Moreover, we discussed the direct product and \( T \)-product of \( T \)-fuzzy subalgebras. We finally showed that the family of \( T \)-fuzzy closed ideals is a completely distributive lattice. These ideas enable us to define the notion of (imaginable) \( T \)-fuzzy filters in BCH-algebras, and to discuss the direct products and \( T \)-products of \( T \)-fuzzy filters. It also gives us possible problems to discuss relations among \( T \)-fuzzy subalgebras, \( T \)-fuzzy closed ideals and \( T \)-fuzzy filters, and to construct the normalizations. We may also use these ideas to introduce the notion of interval-valued fuzzy subalgebras/closed ideals.

Acknowledgement. This work was supported by Korea Research Foundation Grant (KRF-99-015-DP0003).

References


Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: ybjun@nongae.gsnu.ac.kr

Sung Min Hong: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: smhong@nongae.gsnu.ac.kr