SUBGROUPS OF FINITE INDEX IN AN ADDITIVE GROUP OF A RING

DOOSTALI MOJDEH and S. HASSAN HASHEMI

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Abstract. If \( K \) is an infinite field and \( G \subseteq K \) is a subgroup of finite index in an additive group, then \( K^* = G^*G^{*-1} \) where \( G^* \) denotes the set of all invertible elements in \( G \) and \( G^{*-1} \) denotes all inverses of elements of \( G^* \). Similar results hold for various fields, division rings and rings.

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1. Introduction. Let \( R \) be a ring (not necessarily commutative) with a unit element 1 and \( R^* \) denotes the multiplicative group of invertible elements of \( R \). In [9] Leep and Shapiro proved that if \( G \) is a subgroup of index 3 in the multiplicative group \( F^* \), then \( G + G = F \). In [2] Berrizbetia proved that if \( F \) is a field and \( G \subseteq F^* \) is a subgroup of finite index \( n \), then there is a positive integer \( m \), that depends on \( n \), so that if \( \text{char } F = 0 \) or \( \text{char } F \geq m \), then \( G - G = F \). In [1] Bergelson and Shapiro proved that, for various ring \( R \), if \( G \) is a subgroup of finite index of \( R^* \), then \( G - G = R \). In [14] Turnwald proved that if \( G \) is a subgroup of finite index \( n \) in the multiplicative group of a division ring \( F \) then \( G - G = F \) or \( |F| < (n + 1)^2 + 4n \), and if \( |F| > (n - 1)^2 \) and \(-1\) is a sum of elements of \( G \) then every element of \( F \) has this property; the bound \( (n - 1)^2 \) is optimal for infinitely many \( n \). The theories which have important role in studying of the above were Ramsey theory, measure theory and number theory, (cf. [4, 7, 15]). Furthermore in [1] the roles of multiplication and addition were switched, and it was shown that

**Proposition 1.1** (see [1, Proposition 2.14]). Let \( K \) be an infinite field and \( G \subseteq K \) a subgroup of finite index in additive group. Then \( G^*G^{*-1} = K^* \) where \( G^* = G \setminus \{0\} \); that is, for every \( c \in K^* \) there exist \( g_1, g_2 \in G \) such that \( c = g_1 / g_2 \).

**Corollary 1.2.** If \( D \) is an infinite division ring then the above result is satisfied.

In this paper, the roles of multiplication and addition are switched and it is shown that Proposition 1.1 and Corollary 1.2 hold for various fields, division rings and rings.

Now let \( G \subseteq R \) be a subgroup of finite index in an additive group, \( G^* \) be the set of all invertible elements in \( G \), \( G^{*-1} = \{g^{-1} : g \in G^*\} \) and \( G^*G^{*-1} = \{g_1g_2^{-1} : g_1, g_2 \in G^*\} \).

2. \( G^*G^{*-1} \)-ring. Let \( K \) be a ring or field and \( G \subseteq K \) be a subgroup of finite index in an additive group, then it is not necessary that \( G^*G^{*-1} = R^* \) or even \( G^*, G^{*-1} \), and \( G^*G^{*-1} \) have group structure. Note the following statements.

(i) Let \( F = F_p, \) and \( G = F^p \), then \( G^* = F^p_\alpha \), and \( G^*G^{*-1} = F_p^* \neq F^* \).

(ii) Let \( \alpha \) be a root of the polynomial \( x^3 + x + 1 \) over the splitting field \( Z_2(\alpha) = F_8 \),
where \( F_8 \) has 8 elements. Put \( G = \{0, \alpha, \alpha^2, \alpha + \alpha^2\} \) so \( G^* = \{\alpha, \alpha^2, \alpha + \alpha^2\} \). It is clear that \( G^* \) is not a group, because \( G^* \) does not contain the unit element 1. But \( G^* G^{* -1} \) is a subgroup of the multiplicative group \( F_8^* \). Furthermore, \( G^* G^{* -1} = F_8^* \).

(iii) Let \( \beta \) be a root of the polynomial \( x^3 - x + 1 \) over the splitting field \( Z_2(\beta) = F_{16} \). Put \( G = \{0, 1, \beta, 1 + \beta\} \), so \( G^* = \{1, \beta, 1 + \beta\} \), \( G^{* -1} = \{1, \beta^3 + 1, \beta^3 + \beta^2 + \beta\} \) and therefore \( G^* G^{* -1} = \{1, \beta^3 + 1, \beta^3 + \beta^2 + \beta, 1 + \beta^3 + \beta^2 + \beta, 1 + \beta + \beta^2 + \beta^3\} \). It is clear that \( G^* \) is not a group but \( G^* G^{* -1} \) is a proper subgroup of \( F_{16}^* \).

(iv) Let \( R = \mathbb{Z}/n\mathbb{Z} \) where \( n \) is not a prime number. If \( G \subseteq R \) is a proper subgroup in an additive group then it is clear that \( G^* = \emptyset \) and \( G^* G^{* -1} = \emptyset \neq R^* \).

(v) Let \( S = \mathbb{Z}/n\mathbb{Z} \) where \( n \) is a natural number, \( R = S[x] \) and \( H \) is a proper subgroup of \( S \). If \( G = \{f(x) = a_0 + a_1 x + \cdots + a_k x^k : a_i \in S, a_0 \in H\} \), so \( G \subseteq R \) is a subgroup of finite index in an additive group and \( G^* = \emptyset \). If the square of every prime number does not divide \( n \) and \( a_0 \in S \) but \( a_t \in H \), for finitely many \( t > 0 \), then \( G \subseteq R \) is a subgroup of finite index in an additive group, \( G^* = R^* \) and \( G^* G^{* -1} = R^* \).

(vi) Let \( R = \mathbb{Z}/x \mathbb{Z} \) and \( G = \{f(x) = a_0 + a_1 x + \cdots + a_m x^m : a_0 \in m\mathbb{Z}, a_i \in \mathbb{Z} (i \geq 1)\} \). So \( G \subseteq R \) is a subgroup of index \( m \) in an additive group. It is clear that \( G^* = \emptyset = G^* G^{* -1} + R^* = \{1, -1\} \). If for finitely many nonzero indices \( i \)’s, \( a_i \in m_i \mathbb{Z} (m_i > 1) \), then \( G^* = R^* \) and \( G^* G^{* -1} = R^* \).

(vii) Let \( Q \) be the set of rational numbers and \( v_2 \) the 2-adic valuation on \( Q \). Then \( R = \{x \in Q : v_2(x) \geq 0\} = \{m/n \in Q : (n, 2) = 1\} \) is a valuation ring (cf. [3, 10, 11, 12] or [13]). If \( G = \{2m/n \in R : (n, 2) = 1\} \), then \( G \) is a subgroup of finite index 2 in an additive group where \( 0 + G \) and \( 1 + G \) are two distinct cosets in \( R \). It is easy to see that \( G^* = \emptyset, R^* = \{m/n : m = 2k + 1, n = 2l + 1\} \), and \( G^* G^{* -1} \neq R^* \).

The above statement can be shown for any \( p \)-adic valuation ring in \( Q \).

By Proposition 1.1, Corollary 1.2, and the previous statements, the following question may be raised.

**Question 2.1.** If \( F \) is a finite field or a ring and \( G \) is a subgroup of finite index in an additive group, must \( G^* G^{* -1} = F^* ? \)

We will answer the question for all finite fields and some rings.

If \( F \) is a finite field and \(|F|\) is sufficiently large to the index of \( G \), in other words, \( G \) is sufficiently large, then \( G^* G^{* -1} = F^* \).

**Theorem 2.2.** (i) Let \( D \) be a division ring with \( \text{char} D = p \) and \( G \subseteq D \) be a subgroup of index \( p^k \) in an additive group. If \(|D| \geq p^{2k+1} \), then \( G^* G^{* -1} = D^* \).

(ii) If \( G \) is a subgroup of finite index \( n \geq p^k \) in a division ring \( D \) and \(|D| = p^{2k} \), then \( G^* G^{* -1} + D^* \).

**Proof.** (i) Fix \( c \in D^* \). Let the \( g_i \)’s be distinct elements in \( G^* \) (\(|G| > p^{k+1}\)). We form the cosets \((cg_1 + G), \ldots, (cg_{p^k+1} + G)\). By the pigeonhole principle at least two cosets are equal. So \( cg_1 + G = cg_i + G \Rightarrow 0 = c(g_i - g_j) \in G \Rightarrow c(g_i - g_j) = c \Rightarrow g_i - g_j = G^* G^{* -1} \).

(ii) Since \(|D| = p^{2k} \) hence \(|D^*| = p^{2k} - 1 \). By hypothesis \(|D : G| = |D|/|G| \geq p^k \), so \(|G| \leq p^k \) and therefore, \(|G^{* -1}| = |G^*| \geq p^k - 1 \), so we have, \(|G^* G^{* -1}| \leq (p^k - 1)^2 = p^{2k} - 2p^k + 1 < p^{2k} - 1 = |D^*| \) so \( G^* G^{* -1} \neq D^* \).

**Remark 2.3.** Theorem 2.2(ii) gives a bound for \(|D| \) in part (i) which is optimum.
We now give the result which generalizes Proposition 1.1, Corollary 1.2, and Theorem 2.2(i).

**Lemma 2.4.** Let $R$ be a ring and let $S$ be a subset of $R$ with invertible differences, that is, $a - b \in R^*$ for any distinct elements $a, b \in S$.

(i) Suppose $G \subseteq R$ is a subgroup of index $n$ in an additive group. If $|S| > n^2$ then $G^*G^{* - 1} = R^*$.

(ii) If $|S| = \infty$ then $G^*G^{* - 1} = R^*$.

**Proof.** (i) Let $r \in R^*$ be any element. By the pigeonhole principle there exist $s, t \in S$ such that $s - t = a$ and $rs - rt$ both lie in $G^*$. So $r = ba^{-1} \in G^*G^{* - 1}$, as claimed.

(ii) This part is an immediate consequence of part (i).

Apply the lemma with $S = K$ for the proof of Proposition 1.1, with $S = D$ for the proof of Proposition 1.1 and Theorem 2.2(i).

We now state the following definition which is a key concept in the paper. This is the analog of [1, Definition 0.1].

**Definition 2.5.** A ring $R$ is a $G^*G^{* - 1}$-ring, if $G^*G^{* - 1} = R^*$ for every subgroup $G \subseteq R$ of finite index in an additive group.

If $R$ is a ring which is a divisible group, then $R$ has no additive subgroups of finite index (cf. [6]). Combining this statement, Lemma 2.4, and Definition 2.5 we obtain the following result.

**Proposition 2.6.** If $D$ is an infinite division ring, then every ring $R$ which contains a copy of $D$ is a $G^*G^{* - 1}$-ring. In particular if $D = D[[x]]$, $M_n(D)$, $M_n(D[[x]])$, and $M_n(D[[[x]]])$ are $G^*G^{* - 1}$-rings.

**Proof.** If char($D$) is zero then every ring that contains a copy of $D$ is a divisible group and hence $R^* = G^*G^{* - 1}$. If the char($D$) $\neq 0$, Lemma 2.4 implies that $R^* = G^*G^{* - 1}$.

**Remark 2.7.** The converse of Definition 2.5 does not necessarily hold. Let $R = Q[x]$, $G = Q$, then $G^*G^{* - 1} = R^*$. But $G$ is not of finite index in an additive group.

3. Properties of $G^*G^{* - 1}$-ring. In this section, some properties of the $G^*G^{* - 1}$-ring is verified.

**Proposition 3.1.** Let $R$ be a commutative ring and $I$ an ideal of $R$ such that $R/I$ is a $G^*G^{* - 1}$-ring. If $I$ does not contain any additive subgroup of finite index and every element of $1 + I$ is invertible, then $R$ is a $G^*G^{* - 1}$-ring.

**Proof.** Let $G \subseteq R$ be a subgroup of finite index in an additive group. Since $(G + I)/G \cong I/(G \cap I)$ so $|I/(G \cap I)| < \infty$ and hence $I \cap G = I$. Choose $x \in R^*$, then $x + I = (g_1 + I)(g_2 + I)^{-1}$. It is easily seen that $g_1^2, g_2^2 \in G^*$. So $x = g_1^2 + g_2^2 + a$ for some $a \in I$. But $x = (g_1 + a g_2)g_2^{-1}$ where $g_1 + a g_2 \in G^*$, that is, $R^* = G^*G^{* - 1}$.

Let $R$ be a commutative ring and $R[x]$ the polynomial ring over $R$. The element $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ is invertible if and only if $a_0 \in R^*$ and each $a_i$ $(i > 0)$ is nilpotent. So by Proposition 3.1 we have the following result.
Theorem 3.2. Let $R$ be a commutative ring, $R[x]$ the polynomial ring and $I = \{g(x) = a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i (i \geq 1) \text{ is nilpotent}\}$. If $I$ does not contain any additive subgroup of finite index and $R$ is a $G^*G^{-1}$-ring, then $R[x]$ also has that property.

Proof. We have $(R[x]/I)^* = (R/I)^* = \{r + I \mid r \in R^*\}$. Suppose $G/I \subseteq R[x]/I$ is a subgroup of finite index in an additive group, then $G$ is an additive subgroup of finite index in $R[x]$ and $|R/(G \cap R)| = |(R + G)/G| < \infty$. By hypothesis for $r \in R^*$ there exist $g_1, g_2 \in G^* \cap R^*$ such that $r = g_1g_2^{-1}$. So $r + I = g_1g_2^{-1} + I = (g_1 + I)(g_2^{-1} + I) \in (G/I)^*(G/I)^{-1}$, that is, $(R[x]/I)^* = (G/I)^*(G/I)^{-1}$. Now Proposition 3.1 completes the proof.

As an immediate consequence we obtain the following result.

Corollary 3.3. Let $R$ be a commutative ring without any nonzero nilpotent elements. If $R$ is a $G^*G^{-1}$-ring then so is $R[x]$.

The converse of Theorem 3.2 holds in general, see the following result.

Theorem 3.4. Let $R$ be a commutative ring. If $R[x]$ is a $G^*G^{-1}$-ring then $R$ also has that property.

Proof. Let $G \subseteq R$ be a subgroup of finite index in an additive group. Put $H = \{a_0 + r_1x^1 + \cdots + r_kx^k : k \text{ is a nonnegative integer, } a_0 \in G, r_i \in R, i > 0\}$. It is easily seen that, $H \subseteq R[x]$ is a subgroup of finite index in an additive group and $H^* \cap R^* = G^*$. Since $(R[x])^* = H^*H^{-1}$ therefore $R^* = (R[x])^* \cap R^* = (H^*H^{-1}) \cap R^* = G^*G^{-1}$. Thus the proof is complete.

Here, we give a necessary condition for infinite $R; G^*G^{-1} = R^*$, this condition is not sufficient. We also verify the behavior of $G^*G^{-1}$-ring under homomorphisms.

Theorem 3.5. If $R$ is a $G^*G^{-1}$-ring. Assume that $S$ is a nontrivial homomorphic image of $R$ with homomorphism $\varphi : R \rightarrow S$ then

(i) $S$ is infinite.

(ii) Assume $\varphi^{-1}\{1_S\} = \{1_R\}$. If $R^*$ is a $G^*G^{-1}$-ring, then $S^*$ also a $G^*G^{-1}$-ring.

Proof. (i) Suppose $S$ is a finite ring. Let $G = \ker\varphi \cdot G \subseteq R$ is a subgroup of finite index in an additive group, because $R/G \cong S$. Then $G^*G^{-1} = R^*$ therefore $1_R \in G^*G^{-1}$ and $1_S = \varphi(1_R) \in \varphi(G^*G^{-1}) = \varphi(G^*)\varphi(G^{-1}) \subseteq \varphi(G)\varphi(G^{-1}) = 0$ therefore $S = 0$ which is a contradiction, and the proof is complete.

(ii) Let $G \subseteq S$ be a subgroup of finite index in an additive group. Put $H = \varphi^{-1}(G)$, then $H$ is a subgroup of $R$. Now define the following homomorphism

$$\alpha : R \rightarrow \frac{S}{G}, \quad \alpha(x) = \varphi(x) + G,$$  

(3.1)  

so $\alpha$ is also surjective and by the first isomorphism theorem $R/H \cong S/G$. Since $S/G$ is finite then so is $R/H$ and thus $H$ is of finite index. Then by hypothesis $H^*H^{-1} = R^*$ now we have $\varphi(H^*)\varphi(H^{-1}) = \varphi(H^*H^{-1}) = \varphi(R^*) = S^*$ so $G^*G^{-1} = S^*$, and this implies that $S^*$ is a $G^*G^{-1}$-ring.

Theorem 3.5 implies that if $R$ is a finite ring, then $R^*$ is not a $G^*G^{-1}$-ring.
We now verify the behavior of $G^*G^{-1}$-rings under products.

**Theorem 3.6.** Suppose $R = R_1 \times R_2$, if $R_1^*$ and $R_2^*$ is $G^*G^{-1}$-ring then so is $R^*$.

**Proof.** Suppose $G \leq R = R_1 \times R_2$ is a subgroup of finite index of $R$. Put $A_1 = \{a \in R_1 : (a,0) \in G\}$. Now define

$$\alpha : R \rightarrow \frac{R_1 \times R_2}{G}, \quad \alpha(a) = (a,0) + G,$$

so $A_1 = \ker \alpha$. It implies that $A_1 \leq R_1$ is a subgroup of finite index in an additive group. Therefore $A_1^*A_1^{-1} = R_1^*$. Similarly, we define $A_2$ in $R_2$, so $A_2^*A_2^{-1} = R_2^*$. Now we have $A_1 \times A_2 = \{(a,b) | (a,0), (0,b) \in G\} = \{(a,0) + (0,b) | (a,0), (0,b) \in G\} \leq G + G \leq G$ and also $(A_1 \times A_2^*)((A_1^*A_1^{-1})((A_2^*A_2^{-1}) = A_1^*A_1^*A_2^*A_2^{-1} = R_1^* \times R_2^* = R^*$. Since $A_1 \times A_2 \leq G$ then $G^*G^{-1} = R^*$ and thus $R^*$ is a $G^*G^{-1}$-group. □

**Theorem 3.7.** Let $R$ be a ring, $I$ its ideal and every element of $1 + I$ is invertible. If $R$ is $G^*G^{-1}$-ring then $R/I$ is also $G^*G^{-1}$-ring.

**Proof.** Let $G/I \leq R/I$ be a subgroup of finite index in an additive group, then $G \leq R$ is a subgroup of finite index in an additive group. Choose $r + I \in (R/I)^*$ where $r \in R^*$ and $r = g_1g^{-1}_2$ where $g_i \in G^*$, $i = 1,2$. Therefore, $r + I = (g_1 + I)(g_2 + I)^{-1} \in (G/I)^*(G/I)^{-1}$. □

Theorems 3.5, 3.6, the properties of isomorphism, Proposition 2.6, Artin-Wedderburn theorem (cf. [8]), and Theorem 3.7 imply the following result.

**Corollary 3.8.** (i) If $R \cong R_1 \times R_2$ then $R_1^*$ and $R_2^*$ are $G^*G^{-1}$-rings if and only if $R^*$ is a $G^*G^{-1}$-ring.

(ii) Every semisimple ring which has no finite homomorphic image is a $G^*G^{-1}$-ring.

(iii) Let $R$ be a $G^*G^{-1}$-ring and $J$ the Jacobson radical of $R$. Then $S$ is a $G^*G^{-1}$-ring.

**Remark 3.9.** If $S$ is a $G^*G^{-1}$-ring and $R$ is a subring of $S$ then $R$ is not necessarily a $G^*G^{-1}$-ring. So if $\phi : R \rightarrow S$ is a monomorphism and $S$ is a $G^*G^{-1}$-ring then $R$ is not necessarily a $G^*G^{-1}$-ring.

We end this section by verifying whether $D^* = G^*G^{-1}$ $D$ is an infinite division ring and $G = F + [D,D]$ where $F$ denotes the center of $D$ and $[D,D]$ denotes the additive commutator subgroup of $D$, (cf. [5]). As an example see the following example.

**Example 3.10.** Suppose that $D = Q(i,j,k)$ is the rational quaternion, by a simple investigation one can see that $[D,D] = ai + bj + ck$ for $a,b,c \in Q$, therefore $G = F + [D,D] = D$ and so $G^*G^{-1} = D^*$.

This also holds for real quaternions. But in general we have the following result.

This question is answered for a finite-dimensional division algebra (or more generally central algebra).

**Lemma 3.11.** Let $D$ be a finite-dimensional division (or, more generally, central simple) algebra with center $F$. Then $[D,D]$ coincides with the set of elements of $D$ of trace 0.
Proof. Let $d_1, d_2, \ldots, d_n$ be a basis of $D$ of $F$ vector space; here $n = \deg(D)$. Let $T_0$ be the $n^2 - 1$-dimensional of $F$-subspace of $D$ consisting of trace-zero elements. Clearly $[D,D] \subseteq T_0$. Thus it is enough to show that $\dim_F [D,D] \geq n^2 - 1$. Let $K$ be a splitting field of $D$. Then $D \otimes_F K = M_n(K)$. It is easy to see that $[M_n(K), M_n(K)]$ is precisely the set of $n \times n$-matrices of trace zero. On the other hand, this set is spanned by elements $[d_i,d_j]$, as $i, j = 1, 2, \ldots, n^2$. Thus $n^2 - 1$ of these elements are linearly independent over $K$ and, hence, over $F$. This proves the inequality $\dim_F [D,D] \geq n^2 - 1$, as desired.

We therefore conclude that $\dim_F [D,D] = n^2 - 1$, while $\dim_F (D) = n^2$. Thus

$$F + [D,D] = \begin{cases} D, & \text{if char}(F) \text{ does not divide } n, \\ [D,D], & \text{if char}(F) \text{ divides } n. \end{cases}$$

(3.3)

If $D$ is noncommutative (i.e., of degree $n \geq 2$) then the following lemma shows that $D^* = [D,D]^*([D,D])^*-1$ in any characteristic.

Lemma 3.12. Let $D$ be a finite-dimensional division algebra of degree $n \geq 2$ with center $F$ and let $G$ be a $d$-dimensional $F$-vector subspace of $D$.
(a) Assume $2d > n^2$. Then $D^* = G^*G^{*-1}$.
(b) Assume $G = [D,D]$. Then $D^* = G^*G^{*-1}$.

Proof. (a) Let $a \in D^*$. Since $2d > n^2$, the $d$-dimensional $F$-vector spaces $G$ and $aG$ have a nontrivial intersection in $D$, that is, $g_1 = ag_2$ for some $g_1, g_2 \in G^*$. Then $a = g_1g_2^{-1}$, as desired.

(b) By Lemma 3.12, $d = \dim_F [D,D] = n^2 - 1$. Since $D$ is noncommutative, $n \geq 2$ and thus $2d = 2n^2 - 2 > n^2$. Now apply part (a).

Question 3.13. (1) If $R$ is not a finite homomorphic image, must $R^*$ be infinite? Must $R$ contain an infinite subset with invertible differences?
(2) Is there a ring with no finite homomorphic image, but with some finite index subgroup $G$ avoiding all units: $G^* = \emptyset$?
(3) If $R$ is a $G^*G^{*-1}$-ring then must $R^*$ be infinite? If $R$ is a $G^*G^{*-1}$-ring must $R$ contain an infinite sets with invertible differences?
(4) If $R$ is a $G^*G^{*-1}$-ring, then must the matrix ring $M_n(R)$ also have that property? Conversely, if $M_n(R)$ is a $G^*G^{*-1}$-ring, then must $R$ be a $G^*G^{*-1}$-ring?
(5) If $R$ is a $G^*G^{*-1}$-ring and $R$ is a subring of a ring $S$ then must $S$ also be a $G^*G^{*-1}$-ring?
(6) If $R/I$ is a $G^*G^{*-1}$-ring and $1 + I$ is invertible elements then must $R$ also have that property?
(7) Let $D$ be an infinite (algebraic) division algebra over its center $F$. If $G = F + [D,D]$. Is $D^* = G^*G^{*-1}$?

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References


Doostali Mojdeh: Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: dmojdeh@umcc.ac.ir

S. Hassan Hashemi: Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran