REMARKS ON A PAPER BY SILVERMAN

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(Received 15 August 2000 and in revised form 23 November 2000)

ABSTRACT. We improve a result in Silverman’s paper (1999) and answer a question he posed. We also consider a similar problem and obtain sufficient conditions for starlikeness.

2000 Mathematics Subject Classification. 30C45.

1. Introduction. Let $A$ be the class of analytic functions in the unit disc $U = \{z : |z| < 1\}$ having expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (1.1)$$

and let $S \subset A$ be the set of univalent functions in $U$. A function $f \in S$ is said to be starlike of order $\alpha$, $0 < \alpha < 1$, and is denoted by $S^*_\alpha$ if $\Re (f'(z)/f(z)) > \alpha$, $z \in U$ and is said to be convex and is denoted by $C$ if $\Re \{1 + z(f''(z)/f'(z))\} > 0$, $z \in U$.

Silverman [2] investigated properties of the functions $f \in A$ and the class $G_b = \{f \in A : \left|\left|\frac{1 + z(f''(z)/f'(z))}{z(f'(z)/f(z))}\right|\right| - 1 < b, z \in U\}$. (1.2)

Some of the results established by him and relevant to us are given in the following theorem.

**Theorem 1.1.** Let $f \in G_b$ then

(i) If $0 < b \leq 1$, $G_b \subset S^*(2/(1 + \sqrt{1+8b}))$ and in particular $G_1 \subset S^*(1/2)$.

(ii) $G_b \subset C$ for $0 < b \leq 1/\sqrt{2}$ and $G_1 \notin C$.

(iii) For $b \geq 11.66$, $G_b \notin S^*(0)$ and for large enough $b$, $G_b \notin S$.

His method did not extend to $b > 1$ and he expected the order of starlikeness of $G_b$ to decrease from $1/2$ to $0$ as $b$ increases from $1$ to some value $b_0$ after which functions in $G_b$ need not be starlike.

In this paper we establish the following theorems.

**Theorem 1.2.** Let $f \in G_b$, $0 < b \leq 1$, then $G_b \subset S^*(1/(1+b))$ and this order of starlikeness is sharp. Furthermore, for $b > 1$ the elements of $G_b$ need not be regular in $U$.

We notice that if we put $p(z) = z(f'(z)/f(z))$, then $p(z)$ is analytic in $U$ with $p(0) = 1$ and $G_b$ gets transformed to

$$G_b = \left\{f \in A, \left|\left|z \frac{f'(z)}{f(z)}\right|\right| < b, z \in U\right\}.$$ (1.3)
**Definition 1.3.** An analytic function \( f(z) \) is said to be subordinate to another analytic function \( g(z) \), denoted symbolically as \( f(z) \prec g(z) \), if \( f(0) = g(0) \) and there exists an analytic function \( \omega(z) \in A, \omega(0) = 0 \) and \( |\omega(z)| < 1, z \in U \) such that \( f(z) = g(\omega(z)) \).

**Theorem 1.4.** Let \(-1 \leq \alpha \leq 1, 0 \leq a < 1, \lambda > 0\) and let \( p(z) \) be an analytic function in \( U, p(0) = 1, p(z) \neq 0, z \in U \) satisfy the subordination

\[
Z \frac{p'(z)}{p^2(z)} \prec \frac{\lambda z}{(1 + az)^{1+\alpha}}. \tag{1.4}
\]

Then

\[
\frac{1}{p(z)} < 1 - \frac{\lambda}{a\alpha} (1 - (1 + az)^{-\alpha}), \quad \alpha \neq 0,
\]

\[
\Re \frac{1}{p(z)} > 0 \quad \text{if} \quad 0 < \lambda \leq \frac{a\alpha}{1 - (1 + a)^{-\alpha}}, \quad \alpha \neq 0. \tag{1.5}
\]

For \( \alpha = 0 \) and \( 0 < \lambda \leq a / \log(1 + a) \)

\[
p(z) < \frac{1}{1 - (\lambda/a) \log(1 + az)}, \quad \Re p(z) > \left(1 - \frac{\lambda}{d} \log(1 - a)\right)^{-1}. \tag{1.6}
\]

The special case of (1.4) for \( \alpha = 1, \lambda = b - a, -1 \leq b < a \leq 1 \) had been considered in [1]. Silverman’s case corresponds to \( \alpha = -1 \).

In the notation of subordination the class \( G_b \) defined by (1.3) can equivalently be written as

\[
G_b = \left\{ f \in A, \ p(z) = z \frac{f'(z)}{f(z)} \bigg| \ z \frac{p'(z)}{p^2(z)} < b, z \in U \right\}. \tag{1.7}
\]

We need the following result from [3].

**Theorem 1.5.** If \( h \) is starlike in \( U, h(0) = 0 \) and \( p \) is analytic in \( U, p(0) = 1 \) satisfies

\[
z p'(z) < h(z), \tag{1.8}
\]

then

\[
p(z) < q(z) = 1 + \int_0^z \frac{h(t)}{t} dt, \tag{1.9}
\]

where \( q \) is a convex function.

**2. Proof of Theorem 1.2.** From (1.7), \( f \in G_b \) is equivalent to

\[
z \frac{p'(z)}{p^2(z)} = b \omega(z), \quad \omega(0) = 0, \quad |\omega(z)| < 1. \tag{2.1}
\]

By integration from 0 to \( z \) and using \( p(0) = 1 \), we get

\[
\frac{1}{p(z)} = 1 - b \int_0^1 \frac{\omega(tz)}{t} dt. \tag{2.2}
\]
From (2.2) using Schwartz lemma for $\omega(z)$, we get

$$\left|1 - \frac{1}{p(z)}\right| \leq b|z|, \quad (2.3)$$

or equivalently, $|z| = r$ and

$$|p^2(z)| - 2\text{Re}p(z) + 1 \leq b^2r^2|p^2(z)|. \quad (2.4)$$

Therefore, if $b \leq 1$,

$$\left(1 - b^2r^2\right)|p^2(z)| - 2\text{Re}p(z) + 1 \leq 0. \quad (2.5)$$

This is equivalent to

$$\left|p(z) - \frac{1}{1 - b^2r^2}\right| \leq \frac{br}{1 - b^2r^2}, \quad \text{if } 0 \leq b \leq 1, \quad (2.6)$$

and

$$\left|p(z) + \frac{1}{b^2 - 1}\right| \geq \frac{b}{b^2 - 1}, \quad \text{if } b > 1. \quad (2.7)$$

Equation (2.6) gives

$$\text{Re}p(z) \geq \frac{1}{1+b} \quad (2.8)$$

and this is sharp because

$$p(z) = \frac{1}{1+bz} \Rightarrow f(z) = \frac{z}{1+bz} \quad (2.9)$$

satisfies (2.6). The function $p(z)$ given by (2.9) satisfies (1.7) even for $b > 1$. However, (2.9) shows that for $b > 1$ both $p(z)$ and $f(z)$ have a pole at $z = -1/b$ and $\text{Re}p(z)$ can be negative. Thus, the functions $f \in G_b$ for $b > 1$ need not even be regular. \hfill \Box

3. Proof of Theorem 1.4. We notice that the function $z/(1 + az)^{1+\alpha}$, $0 \leq a < 1$, is starlike for $0 < \alpha \leq 1$ and convex for $-1 \leq \alpha \leq 0$. Since every convex function is starlike, we obtain, from (1.4) and Theorem 1.5,

$$\frac{1}{p(z)} < 1 - \frac{\lambda}{a\alpha}(1 - (1+az)^{-\alpha}), \quad \alpha \neq 0, \quad (3.1)$$

and

$$\frac{1}{p(z)} < 1 - \frac{\lambda}{a}\log(1+az), \quad \alpha = 0.$$

As $(1+az)^{-\alpha}$, $|\alpha| \leq 1$, $\alpha \neq 0$, is a convex function with real coefficients, we obtain

$$\text{Re} \frac{1}{p(z)} > 0 \quad \text{if } 0 < \lambda \leq \frac{a\alpha}{1 - (1+a)^{-\alpha}}, \quad |\alpha| \leq 1, \quad \alpha \neq 0, \quad (3.2)$$

and

$$\text{Re} \frac{1}{p(z)} > 0 \quad \text{if } 0 < \lambda \leq \frac{a}{\log(1+a)}, \quad \alpha = 0.$$
Hence,
\[
\text{Re } p(z) \geq \frac{1}{1 - (\lambda/a\alpha)[1 - (1 - a)^{-\alpha}]}, \quad \alpha \neq 0
\]
and \( f(z) \) satisfying \( p(z) = z(f'(z)/f(z)) \) is starlike of order \( 1/(1 - (\lambda/a\alpha)[1 - (1 - a)^{-\alpha}]) \), \( \alpha \neq 0 \) and \( 1/(1 - (\lambda/a)\log(1 - a)) \) for \( \alpha = 0 \).

In the special case \( \alpha = 1 \) and \( \lambda = a - b \) we obtain \( \text{Re } p(z) \geq (1 - a)/(1 - b) \), \( -1 \leq b < a \leq 1 \) which corresponds to the case in [1]. If \( \alpha = -1 \), we obtain \( \text{Re } p(z) > 1/(1 + \lambda) \) which agrees with Theorem 1.2.

References

