ON THE SPECTRUM OF THE DISTRIBUTIONAL KERNEL RELATED TO THE RESIDUE

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ABSTRACT. We study the spectrum of the distributional kernel $K_{\alpha,\beta}(x)$, where $\alpha$ and $\beta$ are complex numbers and $x$ is a point in the space $\mathbb{R}^n$ of the $n$-dimensional Euclidean space. We found that for any nonzero point $\xi$ that belongs to such a spectrum, there exists the residue of the Fourier transform $(-1)^k \hat{K}_{2k,2k}(\xi)$, where $\alpha = \beta = 2k$, $k$ is a nonnegative integer and $\xi \in \mathbb{R}^n$.

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1. Introduction. Gel'fand and Shilov [2, pages 253–256] have studied the generalized function $P_\lambda$, where

\[ P = \sum_{i=1}^p x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \] (1.1)

is a quadratic form, $\lambda$ is a complex number, and $p + q = n$ is the dimension of $\mathbb{R}^n$. They found that $P_\lambda$ has two sets of singularities, namely $\lambda = -1, -2, \ldots, -k, \ldots$ and $\lambda = -n/2, -n/2 - 1, \ldots, -n/2 - k, \ldots$, where $k$ is a positive integer. For the singular point $\lambda = -k$, the generalized function $P_\lambda$ has a simple pole with residue

\[ \left(\frac{-1}{k-1}\right)^k \delta_1^{(k-1)}(P) \quad \text{or} \quad \text{res}_{\lambda=-k} P_\lambda = \left(\frac{-1}{k-1}\right)^k \delta_1^{(k-1)}(P) \] (1.2)

for $p + q = n$ is odd with $p$ odd and $q$ even. Also, for the singular point $\lambda = -n/2 - k$ they obtained

\[ \text{res}_{\lambda=-n/2-k} P_\lambda = \left(\frac{-1}{2^{2k}k!}\right)^{L^k} \delta(n/2 + k) \] (1.3)

for $p + q = n$ is odd with $p$ odd and $q$ even.

Now, let $K_{\alpha,\beta}(x)$ be the convolution of the functions $R^H_\alpha (u)$ and $R^\xi_\beta (v)$, that is,

\[ K_{\alpha,\beta}(x) = R^H_\alpha (u) \ast R^\xi_\beta (v), \] (1.4)

where $R^H_\alpha (u)$ and $R^\xi_\beta (v)$ are defined by (2.1) and (2.3), respectively. Since $R^H_\alpha (u)$ and $R^\xi_\beta (v)$ are tempered distributions, see [4, pages 30–31], thus $K_{\alpha,\beta}(x)$ is also a tempered distribution and is called the distributional kernel.

In this paper, we use the idea of Gel'fand and Shilov to find the residue of the Fourier transform $(-1)^k K_{2k,2k}(\xi)$, where $K_{2k,2k}$ is defined by (1.4) with $\alpha = \beta = 2k$ and $k$ is a nonnegative integer. We found that for any nonzero point $\xi$ that belongs to the spectrum of $(-1)^k K_{2k,2k}(x)$, there exists the residue of the Fourier transform...
\((-1)^k K_{2k,2k}(\xi)\). Actually \((-1)^k K_{2k,2k}(x)\) is an elementary solution of the operator \(\diamond^k\) iterated \(k\) times, that is, \(\diamond^k((-1)^k K_{2k,2k}(x)) = \delta\), where \(\delta\) is the Dirac-delta distribution.

The operator \(\diamond^k\) was first introduced by Kananthai [4] and named as the Diamond operator defined by

\[
\diamond^k = \left[ \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left( \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{n}^2} \right)^2 \right]^k,
\]

where \(p + q = n\) is the dimension of \(\mathbb{R}^n\).

Moreover, the operator \(\diamond^k\) can be expressed as the product of the operators \(\Box^k\) and \(\triangle^k\), that is,

\[
\diamond^k = \Box^k \triangle^k = \triangle^k \Box^k,
\]

where \(\Box^k\) is an ultra-hyperbolic operator iterated \(k\) times defined by

\[
\Box^k = \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k,
\]

where \(p + q = n\). The operator \(\triangle^k\) is an elliptic operator or Laplacian iterated \(k\) times defined by

\[
\triangle^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k.
\]

Trione [7, page 11] has shown that the function \(R_{2k}^\ell(u)\) defined by (2.1) with \(\alpha = 2k\) is an elementary solution of the operator \(\Box^k\). Also, Aguirre Téllez [1, pages 147-148] has proved that the solution \(R_{2k}^\ell(u)\) exists only for odd \(n\) with \(p\) odd and \(q\) even (\(p + q = n\)). Moreover, we can show that the function \((-1)^k R_{2k}^\ell(v)\) is an elementary solution of the operator \(\triangle^k\), where \(R_{2k}^\ell(v)\) is defined by (2.3) with \(\beta = 2k\).

\section{2. Preliminaries}

\textbf{Definition 2.1.} Let \(x = (x_1, x_2, \ldots, x_n)\) be a point of \(\mathbb{R}^n\), and write \(u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2,\ p + q = n\). Denote by \(\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, u > 0\}\) the set of an interior of the forward cone, and \(\Gamma^c\) denotes the closure of \(\Gamma_+\). For any complex number \(\alpha\), define

\[
R_\alpha^\ell(u) = \begin{cases} 
\frac{u^{\alpha-n/2}}{K_\alpha(\alpha)}, & \text{for } x \in \Gamma_+, \\
0, & \text{for } x \notin \Gamma_+, 
\end{cases}
\]

where the constant \(K_\alpha(\alpha)\) is given by the formula

\[
K_\alpha(\alpha) = \frac{\pi^{(n-1)/2}(1/2)}{\Gamma((1-\alpha)/2)\Gamma((\alpha-\alpha)/2)}.
\]

The function \(R_\alpha^\ell(u)\) is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Nozaki [6, page 72]. The function \(R_\alpha^\ell\) is an ordinary function or classical function if \(\text{Re}(\alpha) \geq n\) and is a distribution of \(\alpha\) if \(\text{Re}(\alpha) < n\). Let \(\text{supp} R_\alpha^\ell(u) \subset \Gamma^c\), where \(\text{supp} R_\alpha^\ell(u)\) denotes the support of \(R_\alpha^\ell(u)\).
**Definition 2.2.** Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) be a point of \( \mathbb{R}^n \), and write \( v = x_1^2 + x_2^2 + \cdots + x_n^2 \). For any complex number \( \beta \), define

\[
R^\ell_\beta(v) = \frac{2^{-\beta} \pi^{-n/2} \Gamma((n - \beta)/2) v^{(\beta-n)/2}}{\Gamma(\beta/2)}.
\]

The function \( R^\ell_\beta(v) \) is called the elliptic kernel of Marcel Riesz and is an ordinary function for \( \text{Re}(\beta) \geq n \) and is a distribution of \( \beta \) for \( \text{Re}(\beta) < n \).

**Definition 2.3.** Let \( f \) be a continuous function, then the Fourier transform of \( f \), denoted by \( \hat{f}(\xi) \), is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) dx,
\]

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \), and \( (\xi,x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \). From (2.4), the inverse Fourier transform of \( \hat{f}(\xi) \) is defined by

\[
f(x) = \mathcal{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi) dx.
\]

If \( f \) is a distribution with compact support, by [8, Theorem 7.4.3, page 187] (2.5) can be written as

\[
\hat{f}(\xi) = \langle f(x), e^{-i(\xi,x)} \rangle.
\]

**Lemma 2.4.** Given the equation

\[
\diamondsuit^k u(x) = \delta,
\]

where \( \diamondsuit^k \) is the operator defined by (1.5), and \( \delta \) is the Dirac-delta distribution, \( u(x) \) is an unknown, \( k \) is a nonnegative integer and \( x \in \mathbb{R}^n \), where \( n \) is odd with \( p \) odd, \( q \) even \((n = p + q)\). Then \( u(x) = (-1)^k K_{2k,2k}(x) \) is an elementary solution of the operator \( \diamondsuit^k \). Here \( K_{2k,2k}(x) = R^H_{2k}(u) * R^R_{2k}(v) \) from (1.4) with \( \alpha = \beta = 2k \).

**Proof.** See [4, page 33].

In this paper, we study the spectrum of \((-1)^k K_{2k,2k}(x)\), relate to the residue of the Fourier transform \((-1)^k \hat{K}_{2k,2k}(\xi)\).

**Lemma 2.5.** The Fourier transform

\[
\hat{K}_{\alpha,\beta}(\xi) = (2\pi)^{n/2} \mathcal{F} R^H_{\alpha}(u) \mathcal{F} R^R_\beta(v)
\]

\[
= \frac{(2\pi)^{n/2} K_n(\alpha) H_n(\beta) \Gamma((n - \alpha)/2) \Gamma((n - \beta)/2)}{(2\pi)^{n/2} K_n(\alpha) H_n(\beta) \Gamma((n - \alpha)/2) \Gamma((n - \beta)/2)}
\]

\[
\times \left( \sum_{i=1}^{p} \xi_i - \sum_{j=p+1}^{p+q} \xi_j \right)^{-\alpha} \left( \sum_{i=1}^{n} \xi_i \right)^{-\beta}, \quad i = \sqrt{-1}.
\]


In particular, if $\alpha = \beta = 2k$, $k$ is a nonnegative integer,

$$(-1)^k K_{2k,2k}(\xi) = \frac{1}{(2\pi)^{n/2}} \left( (\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2 \right)^k,$$  \hspace{1cm} (2.9)

where $R_H^H(u)$ and $R_H^F(v)$ are defined by (2.1) and (2.3), respectively.

**Proof.** See [2, page 194] and [5, pages 156–157].

**Definition 2.6.** The spectrum of the distributional kernel $K_{\alpha,\beta}(x)$ is the support of the Fourier transform $\hat{K}_{\alpha,\beta}(\xi)$ or the spectrum of $K_{\alpha,\beta}(x) = \text{supp} \hat{K}_{\alpha,\beta}(\xi)$. Now, from Lemma 2.5 we obtain

$$\text{supp} K_{\alpha,\beta}(x) = (\text{supp} \text{supp} R_H^H(u)) \cap (\text{supp} \text{supp} R_H^F(v)).$$  \hspace{1cm} (2.10)

In particular, from (2.9) the spectrum of

$$(-1)^k K_{2k,2k}(x) = \text{supp} \left[ \frac{1}{(2\pi)^{n/2}} \left( (\sum_{i=1}^p \xi_i^2)^2 - (\sum_{j=p+1}^{p+q} \xi_j^2)^2 \right)^k \right].$$  \hspace{1cm} (2.11)

**Lemma 2.7.** Let $P(x_1, x_2, \ldots, x_n)$ be a quadratic form of positive definite, and is defined by

$$P = P(x_1, x_2, \ldots, x_n) = \left( \sum_{i=1}^p x_i^2 \right)^2 - \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^2,$$  \hspace{1cm} (2.12)

then for any testing function $\varphi(x) \in D$, the space of infinitely differentiable function with compact support,

$$\langle \delta^{(k)}(p), \varphi \rangle = \int_0^{\infty} \left[ \left( \frac{\partial}{4s^3 \partial s} \right)^k \left( \frac{s^{q-4} \varphi(r, s)}{4} \right) \right]_{s=r} r^{p-1} dr,$$  \hspace{1cm} (2.13)

$$\langle \delta^{(k)}(p), \varphi \rangle = (-1)^k \int_0^{\infty} \left[ \left( \frac{\partial}{4r^3 \partial r} \right)^k \left( \frac{r^{p-4} \varphi(r, s)}{4} \right) \right]_{r=s} s^{q-1} ds,$$  \hspace{1cm} (2.14)

where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, and

$$\psi(r, s) = \int \varphi \, d\Omega^p \, d\Omega^q,$$  \hspace{1cm} (2.15)

where $d\Omega^p$ and $d\Omega^q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Both integrals (2.13) and (2.14) converge if $k < (1/4)(p + q - 4)$ for any $\varphi(x) \in D$. If $k \geq (1/4)(p + q - 4)$, these integrals must be understood in the sense of their regularization and (2.13) defined as $\langle \delta^{(k)}_1(p), \varphi \rangle$ and (2.14) defined as $\langle \delta^{(k)}_2(p), \varphi \rangle$. Moreover, if we put $u = r^2$, $v = s^2$, thus (2.13) and (2.14) become

$$\langle \delta^{(k)}(p), \varphi \rangle = \frac{1}{16} \int_0^{\infty} \left[ \frac{\partial^k}{\partial u^{k}} \left( v^{(q-4)/4} \varphi_1(u, v) \right) \right]_{v=u} u^{(1/4)(p-4)} du,$$  \hspace{1cm} (2.16)

$$\langle \delta^{(k)}(p), \varphi \rangle = \frac{(-1)^k}{16} \int_0^{\infty} \left[ \frac{\partial^k}{\partial u^{k}} \left( u^{(p-4)/4} \varphi_1(u, v) \right) \right]_{u=v} v^{(1/4)(q-4)} dv,$$  \hspace{1cm} (2.17)

where $\varphi_1(u, v) = \psi(r, s)$. 
Proof. See [2, pages 247–251].

Lemma 2.8. Let $G_b = \{ \xi \in \mathbb{R}^n : |\xi_1| \leq b_1, |\xi_2| \leq b_2, \ldots, |\xi_n| \leq b_n \}$ be a parallelepiped in $\mathbb{R}^n$ and $b_i (1 \leq i \leq n)$ is a real constant and the inverse Fourier transform of $\hat{K}_{\alpha,\beta}(\xi)$ is defined by

$$K_{\alpha,\beta}(x) = \mathcal{F}^{-1} \hat{K}_{\alpha,\beta}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i\langle \xi, x \rangle} \hat{K}_{\alpha,\beta}(\xi) d\xi,$$

(2.18)

where $K_{\alpha,\beta}$ is defined by (1.4) and $x, \xi \in \mathbb{R}^n$, then $K_{\alpha,\beta}(x)$ can be extended to the entire function $K_{\alpha,\beta}(z)$ and be analytic for all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, where $\mathbb{C}^n$ is the $n$-tuple space of complex number and

$$|K_{\alpha,\beta}(z)| \leq C \exp(b |\text{Im}(z)|),$$

(2.19)

where $\exp(b |\text{Im}(z)|) = \exp[b_1 |\text{Im}(z_1)| + b_2 |\text{Im}(z_2)| + \cdots + b_n |\text{Im}(z_n)|]$ and $C = (1/(2\pi)^{n/2}) \int_{G_b} |\hat{K}_{\alpha,\beta}(\xi)| d\xi$ is a constant. Moreover, $K_{\alpha,\beta}(x)$ has a spectrum contained in $G_b$.

Proof. Since the integral of (2.18) converges for all $\xi \in G_b$, thus $K_{\alpha,\beta}(x)$ can be extended to the entire function $K_{\alpha,\beta}(z)$ and be analytic for all $z \in \mathbb{C}^n$. Thus (2.18) can be written as

$$K_{\alpha,\beta}(z) = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i\langle \xi, z \rangle} K_{\alpha,\beta}(\xi) d\xi.$$  

(2.20)

Now,

$$|K_{\alpha,\beta}(z)| \leq \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\hat{K}_{\alpha,\beta}(\xi)| \exp(i\xi_1 z_1 + i\xi_2 z_2 + \cdots + i\xi_n z_n) \, d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\hat{K}_{\alpha,\beta}(\xi)| \exp(i\xi_1 \sigma_1 + i\xi_2 \sigma_2 + \cdots + i\xi_n \sigma_n - \xi_1 \mu_1 - \xi_2 \mu_2 - \cdots - \xi_n \mu_n) \, d\xi,$$

(2.21)

where

$$z_j = \sigma_j + i\mu_j \quad (j = 1, 2, \ldots, n),$$

(2.22)

thus

$$|K_{\alpha,\beta}(z)| \leq \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\hat{K}_{\alpha,\beta}(\xi)| \, d\xi \exp(b_1 |\mu_1| + b_2 |\mu_2| + \cdots + b_n |\mu_n|)$$

(2.23)

for $|\xi_j| \leq b_j$, or $|K_{\alpha,\beta}(z)| \leq \exp(b_1 |\text{Im}(z_1)| + b_2 |\text{Im}(z_2)| + \cdots + b_n |\text{Im}(z_n)|)$, or

$$|K_{\alpha,\beta}(z)| \leq \exp(b |\text{Im}(z)|),$$

where $C = (1/(2\pi)^{n/2}) \int_{G_b} |\hat{K}_{\alpha,\beta}(\xi)| d\xi$ is a constant. 

We must show that the support of $\hat{K}_{\alpha,\beta}(\xi)$ is contained in $G_b$. Since $K_{\alpha,\beta}(z)$ is an analytic function that satisfies the inequality (2.19) and is called an entire function of order of growth $\leq 1$ and of type $\leq b$, then by Paley-Wiener-Schartz theorem, see [3, page 162], $K_{\alpha,\beta}(\xi)$ has a support contained in $G_b$, that is the spectrum of $K_{\alpha,\beta}(x)$ is contained in $G_b$. 

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In particular, for \( \alpha = \beta = 2k \), the spectrum of \((-1)^k K_{2k,2k}(x)\) is also contained in \(G_b\), that is \(\text{supp}[-(-1)^k K_{2k,2k}(\xi)] \subset G_b\), where \((-1)^k K_{2k,2k}(x)\) is an elementary solution of the Diamond operator \(\diamond^k\) by Lemma 2.4, and the Fourier transform \((-1)^k K_{2k,2k}(\xi)\) given by (2.9) can be defined as follows.

**Definition 2.9.** The Fourier transform

\[
(-1)^k K_{2k,2k}(\xi) = \left\{ \begin{array}{ll}
\frac{1}{(2\pi)^{n/2}} \left[ (\sum_{i=1}^{p} \xi_i^2) - (\sum_{j=p+1}^{q} \xi_j^2) \right]^k, & \text{for } \xi \in G_b, \\
0, & \text{for } \xi \in CG_b,
\end{array} \right.
\]  

(2.24)

where \(\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n\) and \(CG_b\) is the complement of \(G_b\).

3. Main results

**Theorem 3.1.** For any nonzero point \(\xi \in M\) where \(M\) is a spectrum of \((-1)^k K_{2k,2k}(x)\), and \((-1)^k K_{2k,2k}(x)\) is an elementary solution of the operator \(\diamond^k\) by Lemma 2.4. Then there exists the residue of the Fourier transform \((-1)^k K_{2k,2k}(\xi)\) at the singular point \(\lambda = -k\) and such a residue is

\[
\frac{(-1)^k}{(2\pi)^{n/2}(k-1)!} \delta_1^{(k-1)}(p) \quad \text{or} \quad \text{res}_{\lambda=-k}(-1)^k K_{2k,2k}(\xi) = \left( \frac{(-1)^k}{(2\pi)^{n/2}(k-1)!} \right) \delta_1^{(k-1)}(p),
\]

(3.1)

where

\[
P = (\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2) - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2),
\]

(3.2)

\(p + q = n\) and \(\delta_1^{(k-1)}(P)\) is defined by (2.16) with \(\delta_1^{(k-1)}(P) = \delta_1^{(k-1)}(P)\) and \(n\) is odd with \(p\) odd, \(q\) even.

**Proof.** We define the generalized function \(P^\lambda\), where \(P\) is given by (3.2) and \(\lambda\) is a complex number, by

\[
\langle P^\lambda, \varphi \rangle = \int_{P > 0} P^\lambda(\xi) \varphi(\xi) d\xi,
\]

(3.3)

where \(\xi = (\xi_1, \xi_2, \ldots, \xi_n)\) and \(d\xi = d\xi_1 d\xi_2 \cdots d\xi_n\) and \(\varphi(\xi) \in D\), the space of continuous infinitely differentiable function with compact support. Now,

\[
\langle P^\lambda, \varphi \rangle = \int_{P > 0} \left[ (\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2) - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2) \right]^\lambda \varphi(\xi) d\xi.
\]

(3.4)

We transform to bipolar coordinates defined by

\[
\xi_1 = rw_1, \quad \xi_2 = rw_2, \ldots, \quad \xi_p = rw_p,
\]

\[
\xi_{p+1} = sw_{p+1}, \quad \xi_{p+2} = sw_{p+2}, \ldots, \quad \xi_{p+q} = sw_{p+q}, \quad p + q = n,
\]

(3.5)

where \(\sum_{i=1}^{p} w_i^2 = 1\) and \(\sum_{j=p+1}^{p+q} w_j^2 = 1\). Thus

\[
r = \sqrt{\sum_{i=1}^{p} \xi_i^2}, \quad s = \sqrt{\sum_{j=p+1}^{p+q} \xi_j^2}.
\]

(3.6)
We have \( \langle P^\lambda, \varphi \rangle = \int (r^4 - s^4)^\lambda \varphi(\xi) d\xi \). Since the volume \( d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \) where \( d\Omega_p \) and \( d\Omega_q \) are the elements of surface area on the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively. Thus

\[
\langle P^\lambda, \varphi \rangle = \int_{p>0} \int_{p>q} (r^4 - s^4)^\lambda \varphi r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q
\]

where \( \psi(r,s) = \int \varphi d\Omega_p d\Omega_q \).

Since \( \varphi(\xi) \) is in \( D \), then \( \psi(r,s) \) is an infinitely differentiable function of \( r^4 \) and \( s^4 \) with bounded support. We now make the change of variable \( u = r^4, \ v = s^4 \), and writing \( \psi(r,s) = \psi_1(u,v) \). Thus we obtain

\[
\langle P^\lambda, \varphi \rangle = \frac{1}{16} \int_{u=0}^\infty \int_{v=0}^u (u-v)^\lambda \psi_1(u,v) u^{(p-4)/4} v^{(q-4)/4} dv du. \tag{3.8}
\]

Write \( v = ut \). We obtain

\[
\langle P^\lambda, \varphi \rangle = \frac{1}{16} \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \int_0^1 (1-t)^\lambda t^{(q-4)/4} \psi_1(u,ut) dt. \tag{3.9}
\]

Let the function

\[
\Phi(\lambda, u) = \frac{1}{16} \int_0^1 (1-t)^\lambda t^{(q-4)/4} \psi_1(u,ut) dt. \tag{3.10}
\]

Thus \( \Phi(\lambda, u) \) has singularity at \( \lambda = -k \) where it has simple poles. By Gel’fand and Shilov [2, page 254, equation (12)] we obtain the residue of \( \Phi(\lambda, u) \) at \( \lambda = -k \), that is,

\[
\text{res}_{\lambda=-k} \Phi(\lambda, u) = \frac{1}{16} \frac{(-1)^{k-1}}{(k-1)!} \left[ \frac{\partial^{k-1}}{\partial t^{k-1}} (t^{(q-4)/4} \psi_1(u,ut)) \right]_{t=1}. \tag{3.11}
\]

Thus, \( \text{res}_{\lambda=-k} \Phi(\lambda, u) \) is a functional concentrated on the surface \( P = 0 \) \( (t = 1, \ u = v, \ p = u - v = 0) \). On the other hand, from (3.9) and (3.10) we have

\[
\langle P^\lambda, \varphi \rangle = \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi(\lambda, u) du. \tag{3.12}
\]

Thus \( \langle P^\lambda, \varphi \rangle \) in (3.12) has singularities at \( \lambda = -n/4, -n/4 - 1, ..., -n/4 - k \). At these points,

\[
\text{res}_{\lambda=-n/4-k} \langle P^\lambda, \varphi \rangle = \frac{1}{k!} \left[ \frac{\partial^k}{\partial u^k} \Phi \left( -\frac{n}{4} - k, u \right) \right]_{u=0}. \tag{3.13}
\]

Thus the residue of \( \langle P^\lambda, \varphi \rangle \) at \( \lambda = (-1/2)n - k \) is a functional concentrated on the vertex of the surface \( P \). Now consider the case when the singular point \( \lambda = -k \). Write (3.10) in the neighborhood of \( \lambda = -k \) in the form \( \Phi(\lambda, u) = \Phi_0(u)/(\lambda + k) + \Phi_1(\lambda, u) \) where \( \Phi_0(u) = \text{res}_{\lambda=-k} \Phi(\lambda, u) \) and \( \Phi_1(\lambda, u) \) is regular at \( \lambda = -k \). Substitute \( \Phi(\lambda, u) \) into (3.12) we obtain

\[
\langle P^\lambda, \varphi \rangle = \frac{1}{\lambda+k} \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi_0(u) du + \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi_1(\lambda, u) du. \tag{3.14}
\]
Thus \( \text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \int_0^\infty u^{-k+(1/4)(p+q)-1} \Phi_0(u) \, du \). By substituting \( \Phi_0(u) \) and (3.11), we obtain

\[
\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^k}{16(k-1)!} \int_0^\infty \left[ \frac{\partial^{k-1}}{\partial t^{k-1}} \left\{ t^{(q-4)/4} \psi_1(u, ut) \right\} \right]_t=1 u^{-k+(1/4)(p+q)-1} \, du \tag{3.15}
\]

since, we put \( v = ut \). Thus \( \partial^{k-1}/\partial t^{k-1} = u^{k-1}(\partial^{k-1}/\partial v^{k-1}) \), by substituting \( \partial^{k-1}/\partial t^{k-1} \) we obtain

\[
\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^k}{16(k-1)!} \int_0^\infty \left[ \frac{\partial^{k-1}}{\partial t^{k-1}} \left\{ v^{(q-4)/4} \psi_1(u, v) \right\} \right]_{u=v} u^{(1/4)p-1} \, du. \tag{3.16}
\]

Now, by (2.16)

\[
\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(P). \tag{3.17}
\]

Since, by Definition 2.9 we have

\[
(-1)^k K_{2k,2k}^{\hat{\lambda}}(\xi) = \frac{1}{(2\pi)^{n/2}} P^\lambda \quad \text{for} \quad \lambda = -k,
\]

and \( \xi \in G_B \). Let \( M \) be a spectrum of \( (-1)^k K_{2k,2k}(x) \) and \( M \subset G_B \) by Lemma 2.8. Thus for any nonzero \( \xi \in M \) we can find the residue of \((-1)^k K_{2k,2k}(\xi)\), that is,

\[
\text{res}_{\lambda=-k} \langle (-1)^k K_{2k,2k}(\xi), \varphi(\xi) \rangle = \frac{1}{(2\pi)^{n/2}} \text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle
\]

\[
= \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \langle \delta_1^{(k-1)}(P), \varphi \rangle
\]

or \( \text{res}_{\lambda=-k} (-1)^k K_{2k,2k}(\xi) = ((-1)^{k-1}/(2(\pi)^{n/2}(k-1)!)) \delta_1^{(k-1)}(P) \) for \( \xi \in M \) and \( \xi \neq 0 \).

Now consider the case \( \xi = 0 \). We have from (3.13) that, the residue of \( \langle P^\lambda, \varphi \rangle \) occurs at the point \( \lambda = (-1/2)n - k \) that is \( \text{res}_{\lambda=-1/2n} \langle P^\lambda, \varphi \rangle \) is a functional concentrated on the vertex of surface \( P \). Since \( u = 0 \) and \( v = ut \), then \( u = v = 0 \), that implies

\[
\sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_P^2} = \sqrt{\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{P+q}^2} = 0.
\]

It follows that \( \xi_1 = \xi_2 = \cdots = \xi_{p+q} = 0, p + q = n \). Thus, the residue of \( \langle P^\lambda, \varphi \rangle \) is concentrated on the point \( \xi = 0 \).

Since, from Definition 2.9, \( (1/(2\pi)^{n/2}) P^\lambda = (-1)^k K_{2k,2k}(\hat{\xi}) \) if \( \lambda = -k \). Thus we only consider the residue of \( (-1)^k K_{2k,2k}(\hat{\xi}) \) at \( \lambda = -k \). From (3.12), we consider the residue of \( \langle P^\lambda, \varphi \rangle \) only at \( \lambda = -k \). That implies \( (1/4)(p + q) - 1 = 0 \) or \( n = 4 \, (p + q = n) \). Since \( n = 4 \) is an even dimension which contradicts Lemma 2.4, the existence of the elementary solution \((-1)^k K_{2k,2k}(x)\) that exists for odd \( n \). Thus cases (3.12) and (3.13) do not occur. This implies that the case \( \xi = 0 \) does not happen. It follows that

\[
\text{res}_{\lambda=-k} (-1)^k K_{2k,2k}(\xi) = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \delta_1^{(k-1)}(P)
\]

for nonzero point \( \xi \in M \) concentrated on the surface \( P = 0 \), where \( M \) is a spectrum of \((-1)^k K_{2k,2k}(x)\). \( \square \)
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