ON $Q$-ALGEBRAS

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(Received 29 January 2001)

ABSTRACT. We introduce a new notion, called a $Q$-algebra, which is a generalization of the idea of $BCH/BCI/BCK$-algebras and we generalize some theorems discussed in $BCI$-algebras. Moreover, we introduce the notion of “quadratic” $Q$-algebra, and show that every quadratic $Q$-algebra $(X; *, e)$, $e \in X$, has a product of the form $x \ast y = x - y + e$, where $x, y \in X$ when $X$ is a field with $|X| \geq 3$.

2000 Mathematics Subject Classification. 06F35, 03G25.

1. Introduction. Imai and Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras (see [4, 5]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. Neggers and Kim (see [8]) introduced the notion of $d$-algebras, that is, (I) $x \ast x = e$; (IX) $e \ast x = e$; (VI) $x \ast y = e$ and $y \ast x = e$ imply $x = y$, which is another useful generalization of $BCH/BCI/BCK$-algebras, after which they investigated several relations between $d$-algebras and $BCK$-algebras, as well as other relations between $d$-algebras and oriented digraphs. At the same time, Jun, Roh, and Kim [6] introduced a new notion, called a $BH$-algebra, that is, (I) $x \ast x = e$; (II) $x \ast e = x$; (VI) $x \ast y = e$ and $y \ast x = e$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and they showed that there is a maximal ideal in bounded $BH$-algebras. We introduce a new notion, called a $Q$-algebra, which is a generalization of $BCH/BCI/BCK$-algebras and generalize some theorems from the theory of $BCI$-algebras. Moreover, we introduce the notion of “quadratic” $Q$-algebra, and obtain the result that every quadratic $Q$-algebra $(X; *, e)$, $e \in X$, is of the form $x \ast y = x - y + e$, where $x, y \in X$ and $X$ is a field with $|X| \geq 3$, that is, the product is linear in a special way.

2. $Q$-algebras. A $Q$-algebra is a nonempty set $X$ with a constant $0$ and a binary operation “$\ast$” satisfying axioms:

(I) $x \ast x = 0$,

(II) $x \ast 0 = x$,

(III) $(x \ast y) \ast z = (x \ast z) \ast y$ for all $x, y, z \in X$.

For brevity we also call $X$ a $Q$-algebra. In $X$ we can define a binary relation $\leq$ by $x \leq y$ if and only if $x \ast y = 0$. Recently, Ahn and Kim [1] introduced the notion of $QS$-algebras. A $Q$-algebra $X$ is said to be a $QS$-algebra if it satisfies the additional relation:

(IV) $(x \ast y) \ast (x \ast z) = z \ast y$, for any $x, y, z \in X$. 
**Example 2.1.** Let \( \mathbb{Z} \) be the set of all integers and let \( n\mathbb{Z} := \{ nz \mid z \in \mathbb{Z} \} \) where \( n \in \mathbb{Z} \). Then \((\mathbb{Z}; -, 0)\) and \((n\mathbb{Z}; -, 0)\) are \(Q\)-algebras, where “-” is the usual subtraction of integers.

**Example 2.2.** Let \( X := \{0, 1, 2, 3\} \) be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
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<tbody>
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</table>

Then \((X; *, 0)\) is a \(Q\)-algebra, which is not a \(BCH/BCK\)-algebra.

Neggers and Kim [7] introduced the related notion of \(B\)-algebra, that is, algebras \((X; *, 0)\) which satisfy (I) \(x*x = 0\); (II) \(x*0 = x\); (V) \((x*y) * z = x * (z *(0*y))\), for any \(x, y, z \in X\). It is easy to see that \(B\)-algebras and \(Q\)-algebras are different notions. For example, **Example 2.2** is a \(Q\)-algebra, but not a \(B\)-algebra, since \((3*2)*1 = 0 \neq 3 = 3*(1*(0*2))\). Consider the following example. Let \( X := \{0, 1, 2, 3, 4, 5\} \) be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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Then \((X; *, 0)\) is a \(B\)-algebra (see [7]), but not a \(Q\)-algebra, since \((5*3)*4 = 3 \neq 4 = (5*4)*3\).

**Proposition 2.3.** If \((X; *, 0)\) is a \(Q\)-algebra, then
(VII) \((x*(x*y)) * y = 0\), for any \(x, y \in X\).

**Proof.** By (I) and (III), \((x*(x*y)) * y = (x*y) * (x*y') = 0\). \( \square \)

We now investigate some relations between \(Q\)-algebras and \(BCH\)-algebras (also \(BCK/BCI\)-algebras). The following theorems are easily proven, and we omit their proofs.

**Theorem 2.4.** Every \(BCH\)-algebra \(X\) is a \(Q\)-algebra. Every \(Q\)-algebra \(X\) satisfying condition (VI) is a \(BCH\)-algebra.

**Theorem 2.5.** Every \(Q\)-algebra satisfying condition (IV) and (VI) is a \(BCI\)-algebra.
**Theorem 2.6.** Every $Q$-algebra $X$ satisfying conditions (V), (VI), and (VIII) $(x \ast y) \ast x = 0$ for any $x, y \in X$, is a BCK-algebra.

**Theorem 2.7.** Every $Q$-algebra $X$ satisfying $x \ast (x \ast y) = x \ast y$ for all $x, y, z \in X$, is a trivial algebra.

**Proof.** Putting $x = y$ in the equation $x \ast (x \ast y) = x \ast y$, we obtain $x \ast 0 = 0$. By (II) $x = 0$. Hence $X$ is a trivial algebra. □

The following example shows that a $Q$-algebra may not satisfy the associative law.

**Example 2.8.** (a) Let $X := \{0, 1, 2\}$ with the table as follows:

<table>
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<tbody>
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Then $X$ is a $Q$-algebra, but associativity does not hold, since $(0 \ast 1) \ast 2 = 0 \neq 1 = 0 \ast (1 \ast 2)$.

(b) Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all integers and real numbers, respectively. Then $(\mathbb{Z}; -, 0)$ and $(\mathbb{R}; ÷, 1)$ are nonassociative $Q$-algebras where “$-$” is the usual subtraction and “$÷$” is the usual division.

**Theorem 2.9.** Every $Q$-algebra $(X; \ast, 0)$ satisfying the associative law is a group under the operation “$\ast$”.

**Proof.** Putting $x = y = z$ in the associative law $(x \ast y) \ast z = x \ast (y \ast z)$ and using (I) and (II), we obtain $0 \ast x = x \ast 0 = x$. This means that 0 is the zero element of $X$. By (I), every element $x$ of $X$ has as its inverse the element $x$ itself. Therefore $(X; \ast)$ is a group. □

3. The $G$-part of $Q$-algebras. In this section, we investigate the properties of the $G$-part in $Q$-algebras.

**Lemma 3.1.** If $(X; \ast, 0)$ is a $Q$-algebra and $a \ast b = a \ast c$, $a, b, c \in X$, then $0 \ast b = 0 \ast c$.

**Proof.** By (I) and (II) $(a \ast b) \ast a = (a \ast a) \ast b = 0 \ast b$ and $(a \ast c) \ast a = (a \ast a) \ast c = 0 \ast c$. Since $a \ast b = a \ast c$, $0 \ast b = 0 \ast c$. □

**Definition 3.2.** Let $(X; \ast, 0)$ be a $Q$-algebra. For any nonempty subset $S$ of $X$, we define

$$G(S) := \{x \in S \mid 0 \ast x = x\}.$$  

(3.1)

In particular, if $S = X$ then we say that $G(X)$ is the $G$-part of $X$.

**Corollary 3.3.** A left cancellation law holds in $G(X)$.

**Proof.** Let $a, b, c \in G(X)$ with $a \ast b = a \ast c$. By Lemma 3.1, $0 \ast b = 0 \ast c$. Since $b, c \in G(X)$, we obtain $b = c$. □
**Proposition 3.4.** Let \( (X; *, 0) \) be a \( Q \)-algebra. Then \( x \in G(X) \) if and only if \( 0 * x \in G(X) \).

**Proof.** If \( x \in G(X) \), then \( 0 * x = x \) and \( 0 * (0 * x) = 0 * x \). Hence \( 0 * x \in G(X) \). Conversely, if \( 0 * x \in G(x) \), then \( 0 * (0 * x) = 0 * x \). By applying Corollary 3.3, we obtain \( 0 * x = x \). Therefore \( x \in G(X) \).

For any \( Q \)-algebra \( (X; *, 0) \), the set \( B(X) := \{ x \in X \mid 0 * x = 0 \} \) is called the \( p \)-radical of \( X \). If \( B(X) = \{ 0 \} \), then we say that \( X \) is a \( p \)-semisimple \( Q \)-algebra. The following property is obvious.

\((IX)\) \( G(X) \cap B(X) = \{ 0 \} \).

**Proposition 3.5.** If \( (X; *, 0) \) is a \( Q \)-algebra and \( x, y \in X \), then
\[ y \in B(X) \iff (x * y) * x = 0. \tag{3.3} \]

**Proof.** By (I) and (III) \( (x * y) * x = (x * x) * y = 0 * y = 0 \) if and only if \( y \in B(X) \).

**Definition 3.6.** Let \( (X; *, 0) \) be a \( Q \)-algebra and \( I(\neq \emptyset) \subseteq X \). The set \( I \) is called an ideal of \( X \) if for any \( x, y, z \in X \),

\[(1) \ 0 \in I, \]
\[(2) \ x * y \in I \text{ and } y \in I \text{ imply } x \in I. \]

Obviously, \( \{ 0 \} \) and \( X \) are ideals of \( X \). We call \( \{ 0 \} \) and \( X \) the zero ideal and the trivial ideal of \( X \), respectively. An ideal \( I \) is said to be proper if \( I \neq X \).

In Example 2.2 the set \( I := \{ 0, 1, 2 \} \) is an ideal of \( X \).

**Proposition 3.7.** Let \( (X; *, 0) \) be a \( Q \)-algebra. Then \( B(X) \) is an ideal of \( X \).

**Proof.** Since \( (0 * 0) * 0 = 0 \), by Proposition 3.5, \( 0 \in B(X) \). Let \( x * y \in B(X) \) and \( y \in B(X) \). Then by Proposition 3.5, \( ((x * y) * x) * (x * y) = 0 \). By (III), \( ((x * y) * (x * y)) * x = 0 * x = 0 \). Hence \( x \in B(X) \). Therefore \( B(X) \) is an ideal of \( X \).

**Proposition 3.8.** If \( S \) is a subalgebra of a \( Q \)-algebra \( (X; *, 0) \), then \( G(X) \cap S = G(S) \).

**Proof.** It is obvious that \( G(X) \cap S \subseteq G(S) \). If \( x \in G(S) \), then \( 0 * x = x \) and \( x \in S \subseteq X \). Then \( x \in G(X) \) and so \( x \in G(X) \cap S \), which proves the proposition.

**Theorem 3.9.** Let \( (X; *, 0) \) be a \( Q \)-algebra. If \( G(X) = X \), then \( X \) is \( p \)-semisimple.

**Proof.** Assume that \( G(X) = X \). By (X), \( \{ 0 \} = G(X) \cap B(X) = X \cap B(X) = B(X) \). Hence \( X \) is \( p \)-semisimple.

**Theorem 3.10.** If \( (X; *, 0) \) is a \( Q \)-algebra of order 3, then \( |G(X)| \neq 3 \), that is, \( G(X) \neq X \).

**Proof.** For the sake of convenience, let \( X = \{ 0, a, b \} \) be a \( Q \)-algebra. Assume that \( |G(X)| = 3 \), that is, \( G(X) = X \). Then \( 0 * 0 = 0, 0 * a = a, \) and \( 0 * b = b \). From \( x * x = 0 \) and \( x * 0 = x \), it follows that \( a * a = 0, b * b = 0, a * 0 = a, \) and \( b * 0 = b \). Now let \( a * b = 0 \). Then \( 0, a, \) and \( b \) are candidates of the computation. If \( b * a = 0 \), then
implicative, it follows that contradiction. This completes the proof.

Next, if \( b \neq a \), then \( a = b * a = (0 * b) * a = 0 * (a * b) = 0 * b = b = 0 \), a contradiction. For the case \( b = a \), we have \( b = a * b = (0 * b) * a = (0 * a) * b = a * b = 0 \), which is also a contradiction.

Finally, let \( a = b \). If \( a = b \), then \( a = 0 = (b * b) * a = (b * a) * b = b \neq 0 \). This leads to the conclusion that Proposition 2.3 does not hold, a contradiction. Similarly, if \( a = b \), we have \( a = 0 = (b * b) * a = (b * a) * b = b \neq 0 \), which is again a contradiction. This completes the proof.

**Proposition 3.11.** If \((X; *, 0)\) is a \(Q\)-algebra of order 2, then in every case the \(G\)-part \(G(X)\) of \(X\) is an ideal of \(X\).

**Proof.** Let \(|X| = 2\). Then either \(G(X) = \{0\}\) or \(G(X) = X\). In either case, \(G(X)\) is an ideal of \(X\).

**Theorem 3.12.** Let \((X; *, 0)\) be a \(Q\)-algebra of order 3. Then \(G(X)\) is an ideal of \(X\) if and only if \(|G(X)| = 1\).

**Proof.** Let \(X := \{0, a, b\}\) be a \(Q\)-algebra. If \(|G(X)| = 1\), then \(G(X) = \{0\}\) is the trivial ideal of \(X\).

Conversely, assume that \(G(X)\) is an ideal of \(X\). By Theorem 3.10, we know that either \(|G(X)| = 1\) or \(|G(X)| = 2\). Suppose that \(|G(X)| = 2\). Then either \(G(X) = \{0, a\}\) or \(G(X) = \{0, b\}\). If \(G(X) = \{0, a\}\), then \(b * a \in G(X)\) because \(G(X)\) is an ideal of \(X\). Hence \(b * a = b\). Then \(a = 0 = (b * b) * a = (b * a) * b = b * b = 0\), which is a contradiction. Similarly, \(G(X) = \{0, b\}\) leads to a contradiction. Therefore \(|G(X)| \neq 2\) and so \(|G(X)| = 1\).

**Definition 3.13.** An ideal \(I\) of a \(Q\)-algebra \((X; *, 0)\) is said to be implicative if \((x * y) * z \in I\) and \(y * z \in I\), then \(x * z \in I\), for any \(x, y, z \in X\).

**Theorem 3.14.** Let \((X; *, 0)\) be a \(Q\)-algebra and let \(I\) be an implicative ideal of \(X\). Then \(I\) contains the \(G\)-part \(G(X)\) of \(X\).

**Proof.** If \(x \in G(X)\), then \((0 * x) * x = x * x = 0 \in I\) and \(x * x = 0 \in I\). Since \(I\) is implicative, it follows that \(x = 0 * x \in I\). Hence \(G(X) \subseteq I\).

**Definition 3.15.** Let \(X\) and \(Y\) be \(Q\)-algebras. A mapping \(f : X \to Y\) is called a homomorphism if

\[ f(x * y) = f(x) * f(y), \quad \forall x, y \in X. \tag{3.4} \]

A homomorphism \(f\) is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two \(Q\)-algebras \(X\) and \(Y\) are said to be isomorphic, written by \(X \cong Y\), if there exists an isomorphism \(f : X \to Y\). For any homomorphism \(f : X \to Y\), the set \(\{x \in X \mid f(x) = 0\}\) is called the kernel of \(f\), denoted by \(\text{Ker}(f)\) and the set \(\{f(x) \mid x \in X\}\) is called the image of \(f\), denoted by \(\text{Im}(f)\). We denote by \(\text{Hom}(X, Y)\) the set of all homomorphisms of \(Q\)-algebras from \(X\) to \(Y\).
**Proposition 3.16.** Suppose that \( f : X \rightarrow X' \) is a homomorphism of \( Q \)-algebras. Then

1. \( f(0) = 0' \),
2. \( f \) is isotone, that is, if \( x \ast y = 0 \), \( x, y \in X \), then \( f(x) \ast f(y) = 0' \).

**Proof.** Since \( f(0) = f(0 \ast 0) = f(0) \ast f(0) = 0' \), (1) holds. If \( x, y \in X \) and \( x \leq y \), that is, \( x \ast y = 0 \), then by (1), \( f(x) \ast f(y) = f(x \ast y) = f(0) = 0' \). Hence \( f(x) \leq f(y) \), proving (2).

**Theorem 3.17.** Let \((X; \ast, 0)\) and \((X; \ast', 0')\) be \( Q \)-algebras and let \( B \) be an ideal of \( Y \). Then for any \( f \in \text{Hom}(X,Y) \), \( f^{-1}(B) \) is an ideal of \( X \).

**Proof.** By Proposition 3.16(1), \( 0 \in f^{-1}(B) \). Assume that \( x \ast y \in f^{-1}(B) \) and \( y \in f^{-1}(B) \). Then \( f(x) \ast f(y) = f(x \ast y) \in B \). It follows from the fact that \( B \) is an ideal of \( Y \) that \( f(x) \in B \), that is, \( x \in f^{-1}(B) \). This means that \( f^{-1}(B) \) is an ideal of \( X \). The proof is complete.

Since \( \{0'\} \) is an ideal of \( X' \), \( \text{Ker}(f) = f^{-1}(\{0'\}) \) for any \( f \in \text{Hom}(X,Y) \). Hence we obtain the following corollary.

**Corollary 3.18.** The kernel \( \text{Ker}(f) \) is an ideal of \( X \).

4. The quadratic \( Q \)-algebras. Let \( X \) be a field with \(|X| \geq 3\). An algebra \((X; \ast)\) is said to be quadratic if \( x \ast y \) is defined by \( x \ast y := a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \), where \( a_1, \ldots, a_6 \in X \), for any \( x, y \in X \). A quadratic algebra \((X; \ast)\) is said to be quadratic \( Q \)-algebra (resp., \( QS \)-algebra) if it satisfies conditions (I), (II), and (III) (resp., (IV)).

**Theorem 4.1.** Let \( X \) be a field with \(|X| \geq 3\). Then every quadratic \( Q \)-algebra \((X; \ast, e), e \in X, \) has the form \( x \ast y = x - y + e \) where \( x, y \in X \).

**Proof.** Define

\[
x \ast y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F. \tag{4.1}
\]

Consider (I).

\[
e = x \ast x = (A + B + C)x^2 + (D + E)x + F. \tag{4.2}
\]

Let \( x := 0 \) in (4.2). Then we obtain \( F = e \). Hence (4.1) turns out to be

\[
x \ast y = Ax^2 + Bxy + Cy^2 + Dx + Ey + e. \tag{4.3}
\]

If \( y := x \) in (4.3), then

\[
e = x \ast x = (A + B + C)x^2 + (D + E)x + e, \tag{4.4}
\]

for any \( x \in X \), and hence we obtain \( A + B + C = 0 = D + E \), that is, \( E = -D \) and \( B = -A - C \). Hence (4.3) turns out to be

\[
x \ast y = (x - y)(Ax - Cy + D) + e. \tag{4.5}
\]

Let \( y := e \) in (4.5). Then by (II) we have

\[
x = x \ast e = (x - e)(Ax - Ce + D) + e, \tag{4.6}
\]
that is, \((Ax - Ce + D - 1)(x - e) = 0\). Since \(X\) is a field, either \(x - e = 0\) or \(Ax - Ce + D - 1 = 0\). Since \(|X| \geq 3\), we have \(Ax - Ce + D - 1 = 0\), for any \(x \neq e\) in \(X\). This means that \(A = 0, 1 - D + Ce = 0\). Thus (4.5) turns out to be

\[ x \ast y = (x - y) + C(x - y)(e - y) + e. \]  

(4.7)

To satisfy condition (III) we consider \((x \ast y) \ast z\) and \((x \ast z) \ast y\).

\[(x \ast y) \ast z = (x \ast y - z) + C(x \ast y - z)(e - z) + e \]
\[= (x - y - z) + C(x - y)(e - y)(e - z) + 2e \]
\[+ C[(x - y) + C(x - y)(e - y) + (e - z)](e - z) \]  

(4.8)

\[= (x - y - z) + C(x - y)(2e - y - z) + 2e \]
\[+ C^2(x - y)(e - y)(e - z) + C(e - z)^2. \]

Interchange \(y\) with \(z\) in (4.8). Then

\[(x \ast z) \ast y = (x - z - y) + C(x - z)(2e - z - y) + 2e \]
\[+ C^2(x - z)(e - z)(e - y) + C(e - y)^2. \]  

(4.9)

By (4.8) and (4.9) we obtain

\[0 = (x \ast y) \ast z - (x \ast z) \ast y = C^2(e - y)(e - z)(z - y). \]  

(4.10)

Since \(X\) is a field with \(|X| \geq 3\), we obtain \(C = 0\). This means that every quadratic \(Q\)-algebra \((X; \ast, e)\), has the form \(x \ast y = x - y + e\) where \(x, y \in X\), completing the proof. \(\square\)

**Example 4.2.** Let \(\mathbb{R}\) be the set of all real numbers. Define \(x \ast y := x - y + \sqrt{2}\). Then \((\mathbb{R}; \ast, \sqrt{2})\) is a quadratic \(Q\)-algebra.

**Example 4.3.** Let \(\mathcal{F} := \text{GF}(p^n)\) be a Galois field. Define \(x \ast y := x - y + e, e \in \mathcal{F}\). Then \((\mathcal{F}; \ast, e)\) is a quadratic \(Q\)-algebra.

**Theorem 4.4.** Let \(X\) be a field with \(|X| \geq 3\). Then every quadratic \(Q\)-algebra on \(X\) is a (quadratic) \(QS\)-algebra.

**Proof.** Let \((X; \ast, e)\) be a quadratic \(Q\)-algebra. Then \(x \ast y = x - y + e\) for any \(x, y \in X\), and hence

\[(x \ast y) \ast (x \ast z) = (x - y + e) \ast (x - z + e) \]
\[= (x - y + e) - (x - z + e) + e \]
\[= z - y + e = z \ast y, \]  

(4.11)

completing the proof. \(\square\)

**Remark 4.5.** Usually a nonquadratic \(Q\)-algebra need not be a \(QS\)-algebra. See the following example.
**Example 4.6.** Consider the $Q$-algebra $(X; \ast, 0)$ in Example 2.2. This algebra is not a $QS$-algebra, since $(3 \ast 1) \ast (3 \ast 2) = 3 \neq 0 = 2 \ast 1$.

**Corollary 4.7.** Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra on $X$ is a $BCI$-algebra.

**Proof.** It is an immediate consequence of Theorems 2.5 and 4.4.

**Theorem 4.8.** Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra $(X; \ast, e)$ is $p$-semisimple. Furthermore, if $\text{char}(X) \neq 2$, then $G(X) = B(X)$.

**Proof.** Notice that $B(X) = \{ x \in X \mid e \ast x = e \} = \{ x \in X \mid e - x + e = e \} = \{ x \in X \mid e - x = 0 \} = \{ e \}$, that is, $(X; \ast, e)$ is $p$-semisimple. Also, if $\text{char}(X) \neq 2$, then 2 is invertible in $X$ and $G(X) = \{ x \in X \mid e \ast x = x \} = \{ x \in X \mid e - x + e = x \} = \{ x \in X \mid 2e = 2x \} = \{ x \in X \mid e = x \} = \{ e \}$. Of course, if $\text{char}(X) = 2$, then $2e = 0$ for all $x \in X$, whence $G(X) = X$.

This shows that there is a large class of examples of $p$-semisimple $QS$-algebras obtained as quadratic $Q$-algebras.

**Theorem 4.9.** Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra on $X$ is isomorphic to every other such algebra defined on $X$.

**Proof.** Let $x \ast y := x - y + e_1$ and $x \ast' y := x - y + e_2$, where $e_1, e_2 \in X$. Let $\pi(x) := x + (e_2 - e_1)$, for all $x \in X$. Then $\pi(x \ast y) = [(x - y) + e_1] + (e_2 - e_1) = (x - y) + e_2 = (x + (e_2 - e_1)) + (y + (e_2 - e_1)) + e_2 = \pi(x) \ast' \pi(y)$, whence the fact that $\pi^{-1}(x) = x + (e_1 - e_2)$ yields the conclusion that $\pi$ is an isomorphism of $Q$-algebras.

**Theorem 4.10.** Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra $(X; \ast, e)$ determines the abelian group $(X, +)$ via the definition $x + y = x \ast (e - y)$.

**Proof.** Note that $x \ast (e - y) = x - (e - y) + e = x + y$ returns the additive operation of the field $X$, which is an abelian group.

Not every quadratic $Q$-algebra $(X; \ast, e)$, $e \in X$, on a field $X$ with $|X| \geq 3$ need be a $BCK$-algebra, since $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = e + (y - z) \neq e$ in general.

**Problem 4.11.** Construct a cubic $Q$-algebra which is not quadratic. Verify that among such cubic $Q$-algebras there are examples which are not $QS$-algebras. Furthermore, the question whether there are non-$p$-semisimple cubic $Q$-algebras is also of interest.

**References**


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