AN EXTENSION OF A THEOREM OF SAHAB, KHAN, AND SESSA

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Abstract. A fixed point theorem of Fisher and Sessa is generalized to locally convex spaces and the new result is applied to extend a recent theorem on invariant approximation of Sahab, Khan, and Sessa.

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Theorem 1.1. Let $T$ and $I$ be two weakly commuting mappings of a closed convex subset $C$ of a Banach space $X$ into itself satisfying the inequality

$$
\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\},
$$

(1.1)

for all $x, y \in C$, where $a \in (0, 1)$. If $I$ is affine and nonexpansive in $C$ and if $T(C) \subseteq I(C)$, then $T$ and $I$ have a unique common fixed point in $C$.

In this note, we first prove that Theorem 1.1 can appreciably be extended to the setup of a Hausdorff locally convex space. An application of new result is presented to best approximation theory; our work extends earlier results of Brosowski [3], Sahab et al. [12], Singh [14] and many others.

In the sequel, $(E, \tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is said to be an associated family of seminorms for $\tau$ if the family $\{yU : y > 0\}$, where $U = \bigcap_{i=1}^n U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for $\tau$. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is called an augmented associated family for $\tau$ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space $(E, \tau)$, there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [9, page 203]).

The following construction will be crucial. Suppose that $M$ is a $\tau$-bounded subset of $E$. For this set $M$ we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subseteq \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \bigcap_{\alpha=1}^n \lambda_\alpha U_\alpha$ is $\tau$-bounded, $\tau$-closed, absolutely convex and contains $M$. The linear span $E_B$ of $B$ in $E$ is $\bigcup_{n=1}^\infty nB$. The Minkowski functional of $B$ is a norm $\|\cdot\|_B$ on $E_B$. Thus $(E_B, \|\cdot\|_B)$ is a normed space with $B$ as its closed unit ball and $\sup_{\alpha} p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$. 
Following Sessa [13], we say, two selfmaps $I$ and $T$ of a locally convex space $(E, \tau)$ are weakly commuting if and only if

$$p_\alpha(ITx - TIx) \leq p_\alpha(Ix - Tx), \quad (1.2)$$

for each $x \in E$ and $p_\alpha \in A^*(\tau)$. Clearly, commuting maps are weakly commuting but not conversely in general (see [10, 13]). A mapping $T : E \to E$ is said to be nonexpansive on $E$ if $p_\alpha(Tx - Ty) \leq p_\alpha(x - y)$ for all $x, y \in E$ and $p_\alpha \in A^*(\tau)$. The set of fixed points of $T$ on $E$ is denoted by $F(T)$. If $u \in E$, $M \subseteq E$, then for $0 < a \leq 1$, we define the set $D_a$ of best $(M, a)$-approximants to $u$ as follows:

$$D_a = \{ y \in M : ap_\alpha(y - u) = dp_\alpha(u, M), \forall p_\alpha \in A^*(\tau) \}, \quad (1.3)$$

where

$$dp_\alpha(u, M) = \inf \{ p_\alpha(x - u) : x \in M \}. \quad (1.4)$$

Let $D$ denote the set of best approximations to $u$. For $a = 1$, our definition reduces to the set $D$ of best $M$-approximants to $u$. A mapping $T : M \to E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to $x$ and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

2. Results

**Lemma 2.1.** Let $T$ and $I$ be weakly commuting selfmaps of a $\tau$-bounded subset $M$ of a Hausdorff locally convex space $(E, \tau)$. Then $T$ and $I$ are weakly commuting on $M$ with respect to $\|\cdot\|_B$.

**Proof.** By hypothesis for any $x \in M$,

$$p_\alpha(ITx - TIx) \leq p_\alpha(Ix - Tx), \quad \text{for each } p_\alpha \in A^*(\tau). \quad (2.1)$$

Taking supremum on both sides, we get

$$\sup_\alpha p_\alpha \left( \frac{ITx - TIx}{\lambda_\alpha} \right) \leq \sup_\alpha p_\alpha \left( \frac{Ix - Tx}{\lambda_\alpha} \right), \quad (2.2)$$

$$\|ITx - TIx\|_B \leq \|Ix - Tx\|_B \quad \text{as desired.} \qed$$

Note that if $I$ is nonexpansive on a $\tau$-bounded subset $M$ of $E$, then $I$ is also nonexpansive with respect to $\|\cdot\|_B$ (cf. [8, 15]).

We use a technique of Tarafdar [15] to obtain the following common fixed point theorem which generalizes Theorem 1.1 and the main result of Fisher and Sessa [4].

**Theorem 2.2.** Let $M$ be a nonempty $\tau$-bounded, $\tau$-complete, and convex subset of a Hausdorff locally convex space $(E, \tau)$ and $T, I$ two weakly commuting selfmaps of $M$ satisfying the inequality

$$p_\alpha(Tx - Ty) \leq ap_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(Tx - Ix), p_\alpha(Ty - Iy) \}, \quad (2.3)$$

for all $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$ and for some $a \in (0, 1)$. If $I$ is affine and nonexpansive on $M$ and $T(M) \subseteq I(M)$, then $T$ and $I$ have a unique common fixed point.
Proof. Since $M$ is $\tau$-complete, it follows that $(E_B, \| \cdot \|_B)$ is a Banach space and $M$ is complete in it. By Lemma 2.1, $T$ and $I$ are $\| \cdot \|_B$-weakly commuting maps of $M$. From (2.3) we obtain for $x, y \in M$,

$$\sup_{\alpha} p_{\alpha}\left(\frac{Tx - Ty}{\lambda_{\alpha}}\right) \leq a \sup_{\alpha} p_{\alpha}\left(\frac{Ix - Iy}{\lambda_{\alpha}}\right) + (1 - a) \max\left\{\sup_{\alpha} p_{\alpha}\left(\frac{Tx - Ix}{\lambda_{\alpha}}\right), \sup_{\alpha} p_{\alpha}\left(\frac{Ty - Iy}{\lambda_{\alpha}}\right)\right\}.$$  

(2.4)

Thus

$$\|Tx - Ty\|_B \leq a \|Ix - Iy\|_B + (1 - a) \max\{\|Tx - Ix\|_B, \|Ty - Iy\|_B\}. \quad (2.5)$$

It can be shown easily that $I$ is $\| \cdot \|_B$-nonexpansive on $M$. A comparison of our hypothesis with that of Theorem 1.1 tells that we can apply Theorem 1.1 to $M$ as a subset of $(E_B, \| \cdot \|_B)$ to conclude that there exists a unique $a \in M$ such that $a = Ia = Ta$.

An application of Theorem 2.2 establishes the following result in best approximation theory.

**Theorem 2.3.** Let $T$ and $I$ be selfmaps of a Hausdorff locally convex space $(E, \tau)$ and $M$ a subset of $E$ such that $T(\partial M) \subseteq M$, where $\partial M$ denotes boundary of $M$ and $u \in F(T) \cap F(I)$. Suppose that $T$ and $I$ satisfy (2.3) for all $x, y \in D_a = D_a \cup \{u\}$ and $I$ is nonexpansive and affine on $D_a$. For each $p_{\alpha} \in A^*(\tau)$,

$$p_{\alpha}(TIx - ITx) \leq \frac{1}{k} p_{\alpha}((kTx + (1 - k)q) -Ix), \quad (2.6)$$

for all $k \in (0, 1)$, $x \in D_a$ and for some $q \in D_a$. If $D_a$ is nonempty convex, $q \in F(I)$ and $I(D_a) = D_a$, then $I$ and $T$ have a common fixed point in $D_a$ provided one of the following conditions holds:

(i) $D_a$ is $\tau$-compact.

(ii) $D_a$ is weakly compact in $(E, \tau)$, $I$ is weakly continuous and $I - T$ is demiclosed at 0.

Proof. Let $y \in D_a$. Then $Iy \in D_a$, since $I(D_a) = D_a$. Further, if $y \in \partial M$, then $Iy \in M$ for $T(\partial M) \subseteq M$. From (2.3), it follows that for each $p_{\alpha} \in A^*(\tau),$

$$p_{\alpha}(Ty - u) = p_{\alpha}(Ty - Tu) \leq a p_{\alpha}(Iy - Iu) + (1 - a) \max\{p_{\alpha}(Ty - Iy), p_{\alpha}(Tu - Iu)\} \quad (2.7)$$

$$\leq a p_{\alpha}(Iy - u) + (1 - a)(p_{\alpha}(Ty - u) + p_{\alpha}(Iy - u)).$$

So we have, $a p_{\alpha}(Ty - u) \leq p_{\alpha}(Iy - u)$ for all $p_{\alpha} \in A^*(\tau)$. Hence $Ty \in D_a$ which implies that $T$ maps $D_a$ into itself.

Let $\{k_n\}$ be a monotonically nondecreasing sequence of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. Define for each $n \in \mathbb{N}$, a mapping $T_n : D_a \to D_a$ by

$$T_n(x) = k_n Tx + (1 - k_n)q. \quad (2.8)$$

It is possible to define such a mapping $T_n$ for each $n \in \mathbb{N}$, since $D_a$ is convex and
\( q \in D_a \). The map \( I \) is affine so we have
\[ T_nIx = k_nTIx + (1 - k_n)q, \quad IT_nx = k_nITx + (1 - k_n)q. \] (2.9)

From (2.6), it follows that
\[
p_\alpha(T_nIx - IT_nx) = k_n p_\alpha(TIx - ITx) \\
\leq k_n \left( \frac{1}{k_n} \right) p_\alpha(k_nTx + (1 - k_n)q - Tx) \\
= p_\alpha(Tnx - Tx), \quad \forall x \in D_a, p_\alpha \in A_\star(\tau). \] (2.10)

Thus \( I \) and \( T_n \) are weakly commuting on \( D_a \) for each \( n \) and \( T_n(D_a) \subseteq D_a = I(D_a) \).

For all \( x, y \in D_a, p_\alpha \in A_\star(\tau) \) and for all \( j \geq n \), \( (n \text{ fixed}) \), we obtain from (2.3),
\[
p_\alpha(T_nx - T_\alpha y) = k_n p_\alpha(Tx - Ty) \leq k_j p_\alpha(Tx - Ty) \\
\leq p_\alpha(Tx - Ty) \\
\leq ap_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(Tx - IX), p_\alpha(Ty - Iy) \} \\
\leq ap_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(Tx - T_nx) + p_\alpha(T_nx - Ix), \}
\汉语p_\alpha(Ty - T_ny) + p_\alpha(T_ny - Iy) \} \\
\leq ap_\alpha(Ix - Iy) + (1 - a) \max \{ (1 - k_n)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \}
(1 - k_n)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy) \}. \] (2.11)

Hence for all \( j \geq n \), we have
\[
p_\alpha(T_nx - T_\alpha y) \leq ap_\alpha(Ix - Iy) \\
+ (1 - a) \max \{ (1 - k_j)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \}
(1 - k_j)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy) \}. \] (2.12)

As \( \lim k_j = 1 \), from (2.12), for every \( n \in \mathbb{N} \), we have
\[
p_\alpha(T_nx - T_\alpha y) = \lim_j p_\alpha(T_nx - T_\alpha y) \\
\leq \lim_j \{ ap_\alpha(Ix - Iy) + (1 - a) \}
\times \max \{ (1 - k_j)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \}
(1 - k_j)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy) \}. \] (2.13)

This implies that for every \( n \in \mathbb{N} \),
\[
p_\alpha(T_nx - T_\alpha y) \leq ap_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(T_nx - Ix), p_\alpha(T_ny - Iy) \}, \] (2.14)
for all \( x, y \in D_a \) and for all \( p_\alpha \in A_\star(\tau) \).

(i) \( D_a \) being \( \tau \)-compact is \( \tau \)-bounded and \( \tau \)-complete. Thus by Theorem 2.2, for every \( n \in \mathbb{N} \), \( T_n \) and \( I \) have unique common fixed point \( x_n \) in \( D_a \). Now the \( \tau \)-compactness
AN EXTENSION OF A THEOREM OF SAHAB, KHAN, AND SESSA 705

of $D_d$ ensures that \{x_n\} has a convergent subsequence \{x_{n_j}\} which converges to a point $x_o \in D_d$. Since

$$x_{n_j} = T_{n_j}x_{n_j} = k_{n_j}Tx_{n_j} + (1 - k_{n_j})q$$  \hspace{1cm} (2.15)

and $T$ is continuous, so we have, as $j \to \infty$, $Tx_0 = x_0$. The continuity of $I$ implies that

$$Ix_o = I\left(\lim_j x_{n_j}\right) = \lim_j Ix_{n_j} = \lim_j x_{n_j} = x_0.$$  \hspace{1cm} (2.16)

(ii) Weakly compact sets in $(E, \tau)$ are $\tau$-bounded and $\tau$-complete so again by Theorem 2.2, $T_n$ and $I$ have a common fixed point $x_n$ in $D_d$ for each $n$. The set $D_d$ is weakly compact so there is a subsequence \{x_j\} of \{x_n\} converging weakly to some $y \in D_d$. The map $I$ being weakly continuous gives that $Iy = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_jTx_j + (1 - k_j)q$$  \hspace{1cm} (2.17)

implies that $Ix_j - Tx_j = (1 - k_j)[q - Tx_j] \to 0$ as $j \to \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $Iy = Ty = y$. □

**Example 2.4** (cf. MR.89h:54030). Let $M = [1, \infty)$ and $d$ be the absolute value metric on $M$. Define $f$ and $g$ on $M$ by $f(x) = 1 + x$, $g(x) = 1 + 2x$. As $d(fgx, gfx) = 1 \leq x = d(fx, gx)$ for all $x$ in $M$ so $f$ and $g$ are weakly commuting but evidently there exists no sequence \{x_n\} in $M$ for which the condition of compatibility is satisfied ($f$ and $g$ are compatible (see [6]) if $d(fgx_n, gfx_n) \to 0$, as $n \to \infty$, for any sequence \{x_n\} in $M$ satisfying $\lim_n fx_n = \lim_n gx_n = t \in M$).

**Remarks 2.5.** (i) In the light of Example 2.4, the classes of weakly commuting and compatible maps are different and so the statement “weakly commuting maps are compatible” on page 977 in [6] is not valid. Hence Theorem 2.3 cannot be implied by Theorem 5 of Pathak et al. [11] even in Banach space setting.

(ii) Commuting maps satisfy (2.6) so Theorem 2.3(ii) is a proper generalization of the main results of Sahab et al. [12] and Singh [14].

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**References**


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