ON AN APPLICATION OF ALMOST INCREASING SEQUENCES

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ABSTRACT. Using an almost increasing sequence, a result of Mazhar (1977) on |C, 1|k summability factors has been generalized for |C, α; β|k and |N, p; β|k summability factors under weaker conditions.

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1. Introduction. A sequence of (bn) of positive numbers is said to be δ-quasi-monotone, if bn → 0, bn > 0 ultimately and ∆bn ≥ −δn, where (δn) is a sequence of positive numbers (see [2]). Let ∑an be a given infinite series with (sn) as the sequence of its nth partial sums. Let σα n and tα n denote the nth (C, α) means of the sequences (sn) and (nan), respectively, that is,

σα n = 1 / Aα n ∑v=0 n Aα−1 n−v sv,

(1.1)

and

tα n = 1 / Aα n ∑v=1 n Aα−1 n−v va v,

(1.2)

where

Aα n = O (nα), α > −1, α0 = 1, Aα−n = 0, for n > 0.

(1.3)

The series ∑an is said to be summable |C, α|k, k ≥ 1 and α > −1, if (see [6])

∑n=1∞ nβk−1 |σα n − σα n−1| k = ∑n=1∞ 1 / n |tα n| k < ∞,

(1.4)

and it is said to be summable |C, α; β|k, k ≥ 1, α > −1 and β ≥ 0, if (see [7])

∑n=1∞ nβk−1 |σα n − σα n−1| k = ∑n=1∞ nβk−1 |tα n| k < ∞.

(1.5)

Let (pn) be a sequence of positive numbers such that

Pn = ∑n v=0 pv → ∞ as n → ∞, P−i = p−i = 0, i ≥ 1.

(1.6)

The sequence-to-sequence transformation

Tn = 1 / Pn ∑v=0 n pv sv

(1.7)
defines the sequence \((T_n)\) of the Riesz mean or simply the \((\bar{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [8]).

The series \(\sum a_n\) is said to be summable \(|\bar{N}, p_n|_k, k \geq 1\), if (see [3])

\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty, \quad (1.8)
\]

and it is said to be summable \(|\bar{N}, p_n; \beta|_k, k \geq 1, \beta \geq 0\), if (see [4])

\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta k} |\Delta T_{n-1}|^k < \infty, \quad (1.9)
\]

where

\[
\Delta T_{n-1} = -\frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \quad (1.10)
\]

In the special case when \(\beta = 0\) (resp., \(p_n = 1\) for all values of \(n\)), \(|\bar{N}, p_n|_k\) summability is the same as \(|\bar{N}, p_n|_k\) (resp., \(|C, 1; \beta|_k\)) summability.

Also it is known that \(|C, \alpha; \beta|_k\) and \(|\bar{N}, p_n; \beta|_k\) summabilities are, in general, independent of each other.

Mazhar [9] has proved the following theorem for \(|C, 1|_k\) summability factors of infinite series.

**Theorem 1.1** (see [9]). Let \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum n \delta_n \log n < \infty\), \(\sum B_n \log n\) is convergent and \(|\Delta \lambda_n| \leq |B_n|\) for all \(n\). If

\[
\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(\log m) \quad \text{as} \quad m \to \infty, \quad (1.11)
\]

where \((t_n)\) is the \(n\)th \((C, 1)\) mean of the sequence \((n a_n)\), then the series \(\sum a_n \lambda_n\) is summable \(|C, 1|_k, k \geq 1\).

**Remark 1.2.** It should be noted that the condition “\(\sum \delta_n \log n\) is convergent” is enough to prove Theorem 1.1 rather than the conditions “\(\sum \delta_n \log n < \infty\) and \(\sum B_n \log n\) is convergent.”

**2. The main result.** In view of Remark 1.2, the aim of this paper is to generalize Theorem 1.1 for \(|C, \alpha; \beta|_k\) and \(|\bar{N}, p_n; \beta|_k\) summabilities under weaker conditions. For this we need the concept of almost increasing sequence. A positive sequence \((d_n)\) is said to be almost increasing if there exists a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq d_n \leq Bc_n\) (see [1]).Obviously, every increasing sequence is almost increasing but the converse need not be true as can be seen from the example \(d_n = ne^{(-1)^n}\). Since \(\log n\) is increasing, we are weakening the hypotheses of the theorem replacing the increasing sequence by an almost increasing sequence.
Now, we prove the following theorems.

**Theorem 2.1.** Let \((X_n)\) be an almost increasing sequence and \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum nB_nX_n\) convergent and \(|\Delta \lambda_n| \leq |B_n|\) for all \(n\). If the sequence \((u_n^\alpha)\), defined by (see [10])
\[
u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases}
\] (2.1)
satisfies the condition
\[
\sum_{n=1}^{m} n^{\beta k - 1} (u_n^\alpha)^k = O(X_m) \quad \text{as } m \to \infty,
\] (2.2)
then the series \(\sum a_n \lambda_n\) is summable \(|C, \alpha; \beta|_k\), \(k \geq 1\) and \(0 \leq \beta < \alpha \leq 1\).

**Theorem 2.2.** Let \((X_n)\) be an almost increasing sequence and \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum nB_nX_n\) convergent and \(|\Delta \lambda_n| \leq |B_n|\) for all \(n\). If \((p_n)\) is a sequence such that
\[
\sum_{n=v+1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\beta k-1} \frac{1}{p_n^\alpha} = O\left( \left( \frac{P_v}{p_v} \right)^{\beta k} \frac{1}{p_v^\alpha} \right),
\]
\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\beta k-1} |t_n|^k = O(X_m) \quad \text{as } m \to \infty,
\]
\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\beta k-1} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as } m \to \infty,
\]
\[
\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad \text{as } m \to \infty,
\] (2.3)
then the series \(\sum a_n \lambda_n\) is summable \(|\bar{N}, p_n; \beta|_k\) for \(k \geq 1\) and \(0 \leq \beta < 1/k\).

We need the following lemmas for the proof of our theorems.

**Lemma 2.3** (see [5]). If \(0 < \alpha \leq 1\) and \(1 \leq v \leq n\), then
\[
\left| \sum_{p=0}^{v} A_{n-p}^\alpha a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{2.4}
\]

Under the conditions of Theorem 2.2 we obtain the following result.

**Lemma 2.4.** The following equation holds:
\[
|\lambda_n| X_n = O(1) \quad \text{as } n \to \infty. \tag{2.5}
\]
Proof. Since $\lambda_n \to 0$ as $n \to \infty$, we have

$$|\lambda_n| X_n = X_n \left| \sum_{v=n}^{\infty} \Delta \lambda_v \right| \leq X_n \sum_{v=n}^{\infty} |\Delta \lambda_v| \leq \sum_{v=0}^{\infty} X_v |\Delta \lambda_v| \leq \sum_{v=0}^{\infty} X_v |B_v| < \infty. \quad (2.6)$$

Hence $|\lambda_n| X_n = O(1)$ as $n \to \infty$.

3. Proof of Theorem 2.1. Let $(T_\alpha^m)$ be the $n$th $(C, \alpha)$, with $0 < \alpha \leq 1$, mean of the sequence $(na_n \lambda_n)$. Then, by (1.1), we have

$$T_\alpha^m = \frac{1}{A^n} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \quad (3.1)$$

Applying Abel’s transformation, we get

$$T_\alpha^m = \frac{1}{A^n} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A^n} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad (3.2)$$

so that making use of Lemma 2.3, we have

$$|T_\alpha^m| \leq \frac{1}{A^n} \sum_{v=1}^{n-1} \Delta \lambda_v \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \right| + \frac{|\lambda_n|}{A^n} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \leq \frac{1}{A^n} \sum_{v=1}^{n-1} A_{v}^\alpha u_v^\alpha \Delta \lambda_v + \frac{1}{A^n} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v$$

$$= T_{\alpha,1}^m + T_{\alpha,2}^m. \quad (3.3)$$

Since

$$|T_{\alpha,1}^m + T_{\alpha,2}^m|^k \leq 2^k \left( |T_{\alpha,1}^m|^k + |T_{\alpha,2}^m|^k \right), \quad (3.4)$$

to complete the proof of Theorem 2.1, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\beta k-1} |T_{n,r}^\alpha|^k < \infty \quad \text{for } r = 1, 2. \quad (3.5)$$

Now, when $k > 1$, applying Hölder’s inequality with indices $k$ and $k'$, where $1/k + 1/k' = 1$, we get

$$\sum_{n=2}^{m+1} n^{\beta k-1} |T_{n,1}^\alpha|^k \leq \sum_{n=2}^{m+1} n^{\beta k-1} (A_n^\alpha)^{-k} \left( \sum_{v=1}^{n} A_{v}^\alpha u_v^\alpha |B_v| \right)^k$$
\[
\begin{align*}
&\leq \sum_{n=2}^{m+1} n^{\beta k-1} (A_n^\alpha)^{-k} \left\{ \sum_{v=1}^{n-1} (A_v^\alpha)^k (u_v^\alpha)^k \mid B_v \right\} \left\{ \sum_{v=1}^{n-1} |B_v| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\beta k-\alpha k-1} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (u_v^\alpha)^k \mid B_v \right\} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k |B_v| \int_1^{\infty} \frac{dx}{x^{1+\alpha k-\beta k}} \\
&= O(1) \sum_{v=1}^{m} v^{\beta k} (u_v^\alpha)^k |B_v| = O(1) \sum_{v=1}^{m} v |B_v| v^{\beta k-1} (u_v^\alpha)^k \\
&= O(1) \sum_{v=1}^{m} \Delta(v |B_v|) v^{\beta k-1} (u_v^\alpha)^k + O(1) m |B_m| \left| \sum_{v=1}^{m} v^{\beta k-1} (u_v^\alpha)^k \right| \\
&= O(1) \sum_{v=1}^{m} \Delta(v |B_v|) X_v + O(1) m |B_m| X_m \\
&= O(1) \sum_{v=1}^{m} v |B_v| X_v + O(1) \sum_{v=1}^{m-1} (v+1) |B_{v+1}| X_{v+1} + O(1) m |B_m| X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{align*}
\] (3.6)

by virtue of the hypotheses of Theorem 2.1.

Finally, since \(|\lambda_n| = O(1)|B_v|\), by hypothesis

\[
\begin{align*}
\sum_{n=1}^{m} n^{\beta k-1} |T_{n,2}^\alpha| &= \sum_{n=1}^{m} |\lambda_n|^{k-1} n^{\beta k-1} (u_n^\alpha)^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| n^{\beta k-1} (u_n^\alpha)^k \sum_{v=n}^{\infty} |\Delta \lambda_v| \\
&= O(1) \sum_{v=1}^{\infty} |\Delta \lambda_v| \sum_{n=1}^{v} n^{\beta k-1} (u_v^\alpha)^k \\
&= O(1) \sum_{v=1}^{\infty} |B_v| X_v < \infty,
\end{align*}
\] (3.7)

by virtue of the hypotheses of Theorem 2.1.

Therefore, we get

\[
\sum_{n=1}^{m} n^{\beta k-1} |T_{n,r}^\alpha| = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2.
\] (3.8)

This completes the proof of Theorem 2.1. \(\square\)

**Remark 3.1.** It is natural to ask whether our theorem is true with \(\alpha > 1\). All we can say with certainty is that our proof fails if \(\alpha > 1\), for our estimate of \(T_{n,1}^\alpha\) depends upon Lemma 2.3, and Lemma 2.3 is known to be false when \(\alpha > 1\) (see [5] for details).
Proof of Theorem 2.2. Let \((T_n)\) denote the \((\bar{N}, p_n)\) mean of the series \(\sum a_n \lambda_n\). Then, by definition and changing the order of summation, we have

\[
T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{3.9}
\]

Then, for \(n \geq 1\), we have

\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_v a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v a_v. \tag{3.10}
\]

By Abel’s transformation, we have

\[
T_n - T_{n-1} = \frac{n+1}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_v \frac{1}{v} \tag{3.11}
\]

\[
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\]

Since

\[
|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k \left( |T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k \right), \tag{3.12}
\]

to complete the proof of Theorem 2.2, it is enough to show that

\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right)^{\beta k - 1} |T_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3, 4. \tag{3.13}
\]

Since \((\lambda_n) \to 0\) as \(n \to \infty\) by the hypothesis of Theorem 2.2, we have

\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right)^{\beta k - 1} |T_{n,1}|^k = O(1) \sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right)^{\beta k - 1} |\lambda_n|^k |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} |\lambda_n| \left( \frac{p_n}{p_n} \right)^{\beta k - 1} |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \left( \frac{p_v}{p_v} \right)^{\beta k - 1} |t_v|^k
\]

\[
+ O(1) |\lambda_m| \sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right)^{\beta k - 1} |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m
\]

\[
= O(1) \sum_{n=1}^{m-1} |B_n| X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \to \infty,
\]

by virtue of the hypotheses of Theorem 2.2 and in view of Lemma 2.4.
Now, when \( k > 1 \), applying Hölder's inequality with indices \( k \) and \( k' \), where \( 1/k + 1/k' = 1 \), as in \( T_{n,1} \), we have

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k + k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k - 1} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k \right\} \\
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}\\n= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} \left| \lambda_v \right| |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k - 1} \frac{1}{p_{n-1}}\\n= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\beta k - 1} |t_v|^k |\lambda_v| = O(1) \quad \text{as } m \to \infty. 
\]

Again, we have

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k + k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k - 1} \left\{ \sum_{v=1}^{n-1} p_v |B_v| |t_v|^k \right\} \\
\times \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v |B_v| \right\}^{k-1}\\n= O(1) \sum_{v=1}^{m} p_v |B_v| |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\beta k - 1} \frac{1}{p_{n-1}}\\n= O(1) \sum_{v=1}^{m} |B_v| \left( \frac{P_v}{p_v} \right)^{\beta k} \frac{1}{v} |t_v|^k\\n= O(1) \sum_{v=1}^{m} v |B_v| \left( \frac{P_v}{p_v} \right)^{\beta k} \frac{1}{v} |t_v|^k\\n= O(1) \sum_{v=1}^{m-1} \Delta(v |B_v|) \sum_{i=1}^{v} \left( \frac{P_i}{p_i} \right)^{\beta k} \frac{1}{i} |t_i|^k\\n+ O(1) m |B_m| \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\beta k} \frac{1}{v} |t_v|^k\\n= O(1) \sum_{v=1}^{m-1} \Delta(v |B_v|) |X_v| + O(1) m |B_m| |X_m|\\n= O(1) \sum_{v=1}^{m-1} v |X_v| |B_v| + O(1) m \sum_{v=1}^{m-1} (v+1) |B_{v+1}| |X_{v+1}|\\n+ O(1) m |B_m| |X_m|\\n= O(1) \quad \text{as } m \to \infty,
\]

by virtue of the hypotheses of Theorem 2.2.
Finally, we have

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{pn} \right)^{\beta k + k - 1} |T_{n,r}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{pn} \right)^{\beta k - 1} \frac{1}{P_{n-1}} \sum_{n=v+1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k \\
\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} \right\}^{k-1} \\
= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{P_v} \left( \frac{P_v}{P_v} \right)^{\beta k} \frac{1}{v} |t_v|^k \\
= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \left( \frac{P_v}{P_v} \right)^{\beta k - 1} / v |t_v|^k \\
= O(1) \sum_{v=1}^{m} \Delta |\lambda_{v+1}| \sum_{r=1}^{v} \frac{1}{p_r} \left( \frac{P_r}{P_v} \right)^{\beta k - 1} \frac{1}{r} |t_v|^k \\
+ O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{\beta k - 1} \frac{1}{v} |t_v|^k \\
= O(1) \sum_{v=1}^{m} \Delta |\lambda_{v+1}| |X_{v+1}| + O(1) |\lambda_{m+1}| |X_{m+1}| \\
= O(1) \sum_{v=1}^{m-1} |B_v+1| |X_{v+1}| + O(1) |\lambda_{m+1}| |X_{m+1}| \\
= O(1) \text{ as } m \to \infty,
\]

by virtue of the hypotheses of Theorem 2.2 and in view of Lemma 2.4.

Therefore, we get

\[
\sum_{n=1}^{m} \left( \frac{P_n}{pn} \right)^{\beta k + k - 1} |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3, 4. (3.18)
\]

This completes the proof of Theorem 2.2.

If we take \(p_n = 1\) for all values of \(n\) in this theorem, then we get a result concerning the \(|C, 1; \beta|_k\) summability factors.

References


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