A NOTE ON MINIMAL ENVELOPES OF DOUGLAS ALGEBRAS, MINIMAL SUPPORT SETS, AND RESTRICTED DOUGLAS ALGEBRAS

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ABSTRACT. We characterize the interpolating Blaschke products of finite type in terms of their support sets. We also give a sufficient condition on the restricted Douglas algebra of a support set that is invariant under the Bourgain map, and its minimal envelope is singly generated.

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1. Introduction. Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $D$. We denote by $M(H^\infty)$ the set of nonzero complex valued homomorphism of $H^\infty$. With the weak$^*$-topology, $M(H^\infty)$ is a compact Hausdorff space. We identify a function in $H^\infty$ with the Gelfand transform and consider $H^\infty$ the supremum norm closed subalgebra of the space of continuous functions on $M(H^\infty)$. By Carleson’s corona theorem, $D$ is dense in $M(H^\infty)$ in the weak$^*$-topology. For $f \in H^\infty$, put

$$Z(f) = \{x \in M(H^\infty) \setminus D : f(x) = 0\},$$

$$(1.1)$$

$$\{|f| < 1\} = \{x \in M(H^\infty) \setminus D : |f(x)| < 1\}.$$  (1.2)

For two points $x, y$ in $M(H^\infty)$, the pseudohyperbolic distance is given by

$$\rho(x, y) = \sup \{|f(y) : f \in H^\infty, \|f\|_\infty \leq 1, f(x) = 0\}.$$  (1.3)

Then, $0 \leq \rho(x, y) \leq 1$ and put

$$P(x) = \{m \in M(H^\infty) : \rho(x, m) < 1\}.$$  (1.4)

The set $P(x)$ is called the Gleason part containing $x$. For $z, x \in D$, $\rho(z, w) = |(z - w)/(1 - \bar{w}z)|$, and $P(z) = D$. When $P(x) \neq \{x\}$, both $x$ and $P(x)$ are called nontrivial. We denote by $G$ the set of nontrivial points in $M(H^\infty)$.

For an infinite sequence $\{z_n\}_n$ in $D$ with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, the corresponding Blaschke product is defined by

$$b(z) = \prod_{n=1}^\infty \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D.$$  (1.5)

In addition, we have

$$\inf_n (1 - |z_n|^2) |b'(z_n)| > 0,$$  (1.6)
both $b$ and $\{z_n\}_n$ are called interpolating. When $b$ is interpolating and
\[
\lim_{n \to \infty} (1 - |z_n|^2) |b'(z_n)| = 1,
\] (1.6)
both $b$ and $\{z_n\}_n$ are called sparse. An interpolating Blaschke product $b$ is said to be
unimodular on trivial points if $\{x : |b(x)| < 1\} \subseteq G$. In [4], Hoffman proved that for
$x \in M(H^\infty), x \in G$ if and only if $x \in Z(b)$ for some interpolating Blaschke product $b$.
He also proved that for a point $x \in G$, there exists a one-to-one continuous onto map
$L_x : D \to P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map $L_x$,
which is called the Hoffman map for the point $x$, is given by
\[
L_x(z) = \lim_{n \to \infty} \frac{z + z_0}{1 + z_0 z}, \quad z \in D,
\] (1.7)
where $\{Z_\alpha\}_\alpha$ is a net in $D$ which converges to $x$. A part $P(x)$ is called sparse if there is a
sparse Blaschke product $b$ such that $b(x) = 0$. In this case we have $|(b \circ L_x)'(0)| = 1$.
Therefore, $b$ is a sparse Blaschke product if and only if $|(b \circ L_x)'(0)| = 1$ for every
$x \in Z(b)$. A part is called locally sparse if there is an interpolating Blaschke product
$b$ such that $b(x) = 0$ and $|(b \circ L_x)'(0)| = 1$.

For an interpolating Blaschke product $b$ with zeros $\{z_n\}_n$, let
\[
\delta_0(b) = \liminf_{n \to \infty} \min_{Z_n} \rho(z_n, z_k).
\] (1.8)
An interpolating Blaschke product $b$ is called spreading if $\delta_0(b) = 1$. By considering
boundary function, we may consider $H^\infty$, as a closed subalgebra of $L^\infty$, the Banach
algebra of essentially bounded Lebesgue measurable functions on the unit circle $T$.
It is known that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary for $H^\infty$.
Any uniformly closed subalgebra $B$ with $H^\infty \subset B \subset L^\infty$ is called a Douglas algebra.
For a point $x \in M(H^\infty)$, there exists a probability measure $\mu_x$ on $M(L^\infty)$ such that
\[
f(x) = \int_{M(L^\infty)} f \, d\mu_x \quad \forall f \in H^\infty.
\] (1.9)
We denote by $\text{supp} \mu_x$ the closed support set of $\mu_x$. Since $\text{supp} \mu_x$ is a weak peak set of
$M(L^\infty)$ for $H^\infty$, we have $H^\infty_{\text{supp} \mu_x} = \{f \in L^\infty : f_{\text{supp} \mu_x} \in H^\infty_{\text{supp} \mu_x}\}$ is a Douglas algebra.

For $E \subset M(H^\infty)$, a point $x \in E$ is called a minimal support point for $E$ if
\[
\text{supp} \mu_x \subset \text{supp} \mu_y \quad \text{or} \quad \text{supp} \mu_x \cap \text{supp} \mu_y = \emptyset \quad \forall y \in E.
\] (1.10)
If $x$ is a minimal support point for $E$, $\text{supp} \mu_x$ is called a minimal support set for $E$.
For an interpolating Blaschke product $b$, we denote by $m(Z(b))$ the set of minimal
support points for the set $\{x : |b(x)| < 1\}$. Let $X$ be a Banach algebra with identity
and let $B$ be a closed subalgebra of $X$. The Bourgain algebra $B_b$ of $B$ relative to $X$ is
defined by the set of $f$ in $X$ such that $\|ff_n + B\| \to 0$ for every sequence $\{f_n\}_n \in B$ with
$f_n \to 0$ weakly. If $A$ and $B$ are Douglas algebras with $A \subseteq B$ and properly contained,
then $B$ is a minimal superalgebra of $A$ if and only if $\text{supp} \mu_x = \text{supp} \mu_y$ for every
$x, y \in M(A) \setminus M(B)$. We denote by $B_m$ the smallest Douglas algebra which contains all
minimal superalgebras of $B$. We note that $B_b \subset B_m$. An interpolating Blaschke product
$b$ such that $\{x : |b(x)| < 1\} \subseteq G$, with $Z(b) \cap P(x)$ being a finite set for every $x \in Z(b)$,
is said to be of finite type.
2. Proofs of the theorems

**Theorem 2.1.** An interpolating Blaschke product $b$ that is unimodular on trivial parts is of finite type if and only if $m(Z(b)) = \{z : |b(z)| < 1\}$.

**Proof.** Suppose $b$ is an interpolating Blaschke product that is unimodular on the trivial points and of finite type. Let $z \in M(H^\infty + C)$ such that $|b(z)| < 1$. By [1, Theorems 1 and 2], there is an $x \in m(Z(b))$ such that $\text{supp} \mu_x \subset \text{supp} \mu_z$. By [3, Theorem 3.1], the set $\text{supp} \mu_x$ is a maximal support set. Hence $\text{supp} \mu_x = \text{supp} \mu_z$. This implies that $z$ is a minimal support point for $b$, that is, $z \in m(Z(b))$. So $\{z : |b(z)| < 1\} \subset m(Z(b))$. Since $m(Z(b)) \subset \{z : |b(z)| < 1\}$, we have $\{z : |b(z)| < 1\} = m(Z(b))$. Conversely, suppose $m(Z(b)) = \{z : |b(z)| < 1\}$ and assume that $b$ is unimodular on trivial points but not of finite type. Then there is a $y \in Z(b)$ such that the set $Z(b) \cap P(y)$ is an infinite set. By [2, Theorems 1 and 2], there is an $x \in M(H^\infty + C)$ such that $|b(x)| < 1$, an uncountable index set $I$ such that for $\alpha, \beta \in I$, $\alpha \neq \beta$, $\text{supp} \mu_{x_\alpha} \cap \text{supp} \mu_{x_\beta} = \phi$, $x_\alpha, x_\beta \in m(Z(b))$, and $\text{supp} \mu_{x_\alpha} \subset \text{supp} \mu_x$ for all $\alpha \in I$. Since $\text{supp} \mu_{x_\alpha}$ is properly contained in $\text{supp} \mu_x$, this implies that $x \notin m(Z(b))$ but $|b(x)| < 1$. This contradicts our assumption that $\{z : |b(z)| < 1\} = m(Z(b))$. Thus, $b$ is of finite type.

**Theorem 2.2.** Suppose that $b$ is a spreading nonsparse Blaschke product, and $x \in m(Z(b))$ such that $|(b \circ L_x)'(0)| \neq 1$. Then

(i) $\left( H^\infty_{\text{supp} \mu_x} \right) b = H^\infty_{\text{supp} \mu_x}$,

(ii) $\left( H^\infty_{\text{supp} \mu_x} \right) m = H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right]$.

**Proof.** By [5, Lemma 2.1], we have that $P_x$ is a nonlocally sparse part. Hence, by [6, Theorem 5] we have that (i) holds.

Since $b$ is spreading and $x \in m(Z(b))$,

$$M \left( H^\infty_{\text{supp} \mu_x} \right) = M \left( H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right] \right) \cup E_x,$$

where $E_x = \{y \in M(H^\infty + C) : \text{supp} \mu_x = \text{supp} \mu_y\}$. This implies that $H^\infty_{\text{supp} \mu_x}$ is properly contained in $(H^\infty_{\text{supp} \mu_x})_m$. Since $H^\infty_{\text{supp} \mu_x}$ is a maximal subalgebra of $H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right]$, $H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right]$ is contained in $(H^\infty_{\text{supp} \mu_x})_m$. Since

$$M \left( H^\infty_{\text{supp} \mu_x} \right) = M(L^\infty) \cup \{y \in M(H^\infty + C) : \text{supp} \mu_y \subseteq \text{supp} \mu_x\},$$

we show that if $q$ is an interpolating Blaschke product such that $\hat{q} \in (H^\infty_{\text{supp} \mu_x})_m$, then $H^\infty_{\text{supp} \mu_x} \left[ \hat{q} \right] = H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right]$. This proves (ii). Suppose that we have $H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right]$ properly contained in $(H^\infty_{\text{supp} \mu_x})_m$, then we have $M((H^\infty_{\text{supp} \mu_x})_m)$ properly contained in $M(H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right])$. So there is a $y \in M(H^\infty_{\text{supp} \mu_x} \left[ \hat{b} \right])$, an interpolating Blaschke product $q$ with $\hat{q} \in (H^\infty_{\text{supp} \mu_x})_m$ and $q(y) = 0$. By (2.2) we have $y \in M(H^\infty_{\text{supp} \mu_x})$ but $y \notin E_x$. Again, by (2.2), this implies that $\text{supp} \mu_y$ properly contained in the $\text{supp} \mu_x$. By [2, Theorems 1 and 2], there is an uncountable index set $I$ such that if $\alpha, \beta \in I$, $\alpha \neq \beta$, there are $x_\alpha, x_\beta \in Z(q)$ with $\text{supp} \mu_\alpha \cap \text{supp} \mu_{x_\beta} = \phi$ and $\text{supp} \mu_\alpha, \text{supp} \mu_{x_\beta}$ are both properly contained in $\text{supp} \mu_x$. This implies that

$$\bigcup_{\alpha \in I} E_x \subset \left\{ m \in M(H^\infty_{\text{supp} \mu_x}) : |q(m)| < 1 \right\},$$

for all $\alpha \neq \beta$.
But this contradicts [2, Theorem 3] since $\alpha \neq \beta$ implies that $E_{X_\alpha} \cap E_{X_\beta} = \emptyset$. Thus, no such $y$ exists and we have $H_{\supp x}^{\infty} [\hat{b}] = H_{\supp x}^{\infty} [\hat{q}]$. So (ii) holds. \hfill \Box

References


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