STABLE RINGS GENERATED BY THEIR UNITS

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ABSTRACT. We introduce the class of rings satisfying \((m, 1)\)-stable range and investigate equivalent characterizations of such rings. These give generalizations of the corresponding results by Badawi (1994), Ehrlich (1976), and Fisher and Snider (1976).

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Let \(R\) be an associative ring with identity. A ring \(R\) is said to have stable range one provided that \(aR + bR = R\) implies that \(a + by \in U(R)\) for \(y \in R\). It is well known that \(M_R\) cancels from direct sums if \(\text{End}_R M\) has stable range one. For further properties of stable range one condition, we refer the reader to \([1, 2, 5, 7, 9, 10, 13, 14]\).

Many authors have studied rings generated by their units (see \([3, 4, 7, 8, 10, 12]\)). It was shown that every unit-regular ring in which 2 is invertible is generated by its unit (see \([7, \text{Theorem 5}]\)) and every strongly \(\pi\)-regular ring in which 2 is invertible is generated by its units (see \([8, \text{Theorem 3}]\)). So far one always investigate such rings under stable range one condition.

In this paper, we generalize stable range one condition and introduce rings satisfying \((m, 1)\)-stable range so as to investigate rings generated by their units. Also we give generalizations of the corresponding results in \([3, 7, 8]\).

Throughout, rings are associative with identity and modules are right modules. \(\text{GL}_n(R)\) denotes the general linear group of \(R\), \(U(R)\) denotes the set of units of \(R\), and that \(U_m(R) = \{x \in R \mid 3u_1, \ldots, u_m \in U(R) \text{ such that } x = u_1 + \cdots + u_m\}\). Let \(B_{ij}(x) = I_2 + xe_{ij}\) (\(i \neq j, 1 \leq i, j \leq 2\)), \([\alpha, \beta] = \alpha e_{11} + \beta e_{22}\), where \(e_{ij}\) (\(1 \leq i, j \leq 2\)) are matrix units (1 in the \(i, j\) position and 0 elsewhere).

**Definition 1.** The ring \(R\) is said to satisfy \((m, 1)\)-stable range provided that \(aR + bR = R\) implies that \(a + by \in U(R)\) for \(y \in U_m(R)\).

**Proposition 2.** The following are equivalent:
1. The ring \(R\) satisfies \((m, 1)\)-stable range.
2. Whenever \(ax + b = 1\), there exists \(y \in U_m(R)\) such that \(a + by \in U(R)\).

**Proof.** (1)\(\Rightarrow\) (2). The proof is obvious.

(2)\(\Rightarrow\) (1). Given \(aR + bR = R\), then \(ax + by = 1\) for some \(x, y \in R\). So we can find \(z \in U_m(R)\) such that \(axz + b = u \in U(R)\), and then \(axzu^{-1} + bu^{-1} = 1\). Hence we have \(w \in U_m(R)\) such that \(a + bu^{-1}w \in U(R)\). Clearly, \(u^{-1}w \in U_m(R)\), as desired. \(\Box\)

**Proposition 3.** The following are equivalent:
1. The ring \(R\) satisfies \((m, 1)\)-stable range.
2. The ring \(R/J(R)\) satisfies \((m, 1)\)-stable range.
\textbf{Proof.} \((1) \Rightarrow (2).\) Given \(a \hat{x} + \hat{b} = \hat{1} \in R/J(R),\) then \(ax + (b + r) = 1\) for some \(r \in J(R).\) Since \(R\) satisfies \((m, 1)\)-stable range, we have \(y \in U_m(R)\) such that \(a + (b + r)y \in U(R).\) Therefore \(\hat{a} + \hat{b} \hat{y} \in U(R/J(R))\) with \(\hat{y} \in U_m(R/J(R)).\) Hence \(R/J(R)\) satisfies \((m, 1)\)-stable range by Proposition 2.

\((2) \Rightarrow (1).\) Given \(ax + b = 1 \in R,\) then \(a \hat{x} + \hat{b} = \hat{1} \in R/J(R).\) So there is \(\hat{y} \in U_m(R/J(R))\) such that \(\hat{a} + \hat{b} \hat{y} = \hat{a} \in U(R/J(R)).\) Assume that \(y = w_1 + w_2 + \cdots + w_m\) with all \(w_i \in U(R/J(R)).\) Since units lift modulo \(J(R),\) we may assume that all \(w_i \in U(R)\) and \(u \in U(R),\) and that \(a + b(w_1 + w_2 + \cdots + w_m) = u + r\) for some \(r \in J(R).\) Obviously, \(u + r \in U(R)\) and \(w_1 + w_2 + \cdots + w_m \in U_m(R).\) Hence \(R\) satisfies \((m, 1)\)-stable range, as asserted. \(\square\)

\textbf{Theorem 4.} Let \(R\) be an associative ring with identity, \(K\) a set of some elements of \(R.\) Then the following are equivalent:

1. Whenever \(ax + b = 1,\) there exists \(y \in K\) such that \(a + by \in U(R).\)
2. Whenever \(ax + b = 1,\) there exists \(z \in K\) such that \(x + zb \in U(R).\)

\textbf{Proof.} \((1) \Rightarrow (2).\) Since \(ax + b = 1,\) we see that \(\left(\frac{a-b}{1-x}\right)^{-1} = \left(1 - x\frac{1}{a}\right) \in \text{GL}_2(R).\)

Clearly, \(xa + (1 - xa) = 1.\) So there exists \(z \in K\) such that \(x + (1 - xa)z = u \in U(R).\)

Hence \(\left(\frac{a-b}{1-x}\right)^{-1} = \left(\frac{u}{u^*} \frac{x^*}{1}\right) \in \text{GL}_2(R).\) Thus we know that \(\left(\frac{a-b}{1-x}\right)^{-1} = \left(\frac{u}{u^*} \frac{x^*}{1}\right).

Therefore \(\left(\frac{a-b}{x}\right)^{-1} = \left(\frac{1}{1}\right) \left(\frac{u}{u^*} \frac{x^*}{1}\right) = \left(\frac{u}{u^*} \frac{x^*}{1}\right)^{-1} = \left(\frac{u}{u^*} \frac{x^*}{1}\right).

Hence \(\left(\frac{a-b}{x}\right)^{-1} = \left(\frac{x}{x^*} \frac{u}{u^*} \frac{x^*}{1}\right),\) \(v \in U(R).\) So \(\left(\frac{1}{1}\right) \left(\frac{a-b}{x}\right)^{-1} = \left(\frac{x}{x^*} \frac{u}{u^*} \frac{x^*}{1}\right)^{-1} = \left(\frac{x}{x^*} \frac{u}{u^*} \frac{x^*}{1}\right).

Thus, we see that \(x + zb = v \in U(R),\) as required.

\((2) \Rightarrow (1).\) Applying \((1) \Rightarrow (2)\) to the opposite ring \(R^{op},\) we complete the proof. \(\square\)

Theorem 4 is a general result for symmetry of stable range conditions. As applications, we see that stable range one conditions, unit 1-stable range conditions and rings having many unit-regular elements are symmetric. The following result shows that \((m, 1)\)-stable range condition is right-left symmetric.

\textbf{Corollary 5.} The following are equivalent:

1. The ring \(R\) satisfies \((m, 1)\)-stable range.
2. Whenever \(ax + b = 1,\) there exists some \(z \in U_m(R)\) such that \(x + zb \in U(R).\)
3. Whenever \(Ra + Rb = R,\) there exists some \(z \in U_m(R)\) such that \(a + zb \in U(R).\)

\textbf{Proof.} \((1) \Leftrightarrow (2).\) Set \(K = U_m(R).\) Then the equivalence follows by Theorem 4.

\((3) \Rightarrow (2).\) The proof is trivial.

\((2) \Rightarrow (3).\) Given \(Ra + Rb = R,\) then \(xa + yb = 1\) for some \(x, y \in R.\) So we have \(s \in U_m(R)\) such that \(sxa + b = u \in U(R),\) hence \(u^{-1}sxa + u^{-1}b = 1.\) Therefore \(a + vu^{-1}b \in U(R)\) for some \(v \in U_m(R),\) as required. \(\square\)

\textbf{Proposition 6.} The following are equivalent:

1. The ring \(R\) satisfies \((m, 1)\)-stable range.
2. For any \(A \in \text{GL}_2(R),\) there exists some \(w \in U_m(R)\) such that \(A = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(\ast).\)
3. For any \(A \in \text{GL}_2(R),\) there exists some \(w \in U_m(R)\) such that \(A = [\ast, \ast]B_{12}(\ast)B_{21}(\ast)B_{12}(w).\)
PROOF. (1)⇒(2). Let $A \in \text{GL}_2(R)$, and let $A^{-1} = (b_{ij})$. Since $b_{11}R + b_{12}R = R$, we can find some $y \in U_m(R)$ such that $b_{11} + b_{12}y = u \in U(R)$. We easily check that $A^{-1} = B_{21}(b_{11} + b_{12}u^{-1})[u, b_{22} - (b_{11} + b_{12}y)u^{-1}b_{12}]B_{12}(u^{-1}b_{12})B_{21}(-y)$. Thus $A = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(\ast)$ for some $w \in U_m(R)$.

(2)⇒(1). Given $ax + b = 1$ in $R$, then we have $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in \text{GL}_2(R)$. Thus we have a $w \in U_m(R)$ such that $(a\ast b^{-1}) = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(\ast)$. Therefore we see that $(\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [\ast, \ast]B_{21}(\ast)B_{12}(\ast)B_{21}(-y)$ for some $y \in U_m(R)$. Consequently, $a + by \in U(R)$ with $y \in U_m(R)$, as desired.

(1)⇔(3). Applying (1)⇔(2) to the opposite ring $R^{op}$, we complete the proof by the symmetry of $(m, 1)$-stable range conditions. □

Let $R$ be generated by $m$ units. If $R$ has stable range one, then it satisfies $(m, 1)$-stable range. Conversely, we easily check that every ring satisfying $(m, 1)$-stable range is generated by $m + 1$ units. Now we show that $(m, 1)$-stable range condition is inherited by matrix rings.

**Lemma 7.** The following are equivalent:

(1) The ring $R$ satisfies $(m, 1)$-stable range.

(2) Given $ax + b = 1$ in $R$, then there exists $y \in R$ such that $a + by \in U(R)$ and $1 - xy \in U_m(R)$.

(3) Given $ax + b = 1$ in $R$, then there exists $z \in R$ such that $x + zb \in U(R)$ and $1 - za \in U_m(R)$.

**Proof.** (1)⇒(2). Given $ax + b = 1$ in $R$, then $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in \text{GL}_2(R)$. In view of Proposition 6, we have a $w \in U_m(R)$ such that $(\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(\ast)$. So we can find some $-y \in R$ such that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(-y)$. Therefore $a + by \in U(R)$ and $1 - xy = -(1 + xy) \in U_m(R)$, as required.

(2)⇒(3). Given $ax + b = 1$ in $R$, then there exists some $y \in R$ such that $a + by = u \in U(R)$ and $1 - xy = v \in U_m(R)$. So we know that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(\ast)$ for some $w \in U_m(R)$. Thus $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [\ast, \ast]B_{21}(w)B_{12}(\ast)B_{21}(-y)$. So we can find $z \in U_m(R)$ such that $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = [\ast, \ast]B_{21}(\ast)B_{21}(\ast)$. Consequently, we show that $x + zb \in U(R)$ for some $z \in U_m(R)$. Therefore $R$ satisfies $(m, 1)$-stable range by Corollary 5.

(1)⇔(3). Applying (1)⇔(2) to the opposite ring $R^{op}$, we complete the proof. □

In [6], the author shows that every matrix ring over a ring satisfying unit 1-stable range also satisfies unit 1-stable range. Now we extend [6, Theorem 2.2] to $(m, 1)$-stable range conditions by a similar route.

**Theorem 8.** If $R$ satisfies $(m, 1)$-stable range, then so does $M_n(R)$ for any $n \geq 1$.

**Proof.** Given $BC + D = I_n$ in $M_n(R)$, then $A = (\left(\begin{smallmatrix} B & D \\ I_n & C \end{smallmatrix}\right) \in \text{GL}_{2n}(R)$. Set $A = (A_{ij}) (1 \leq i, j \leq 2)$ with all $A_{ij} = (A_{ij}^{(t)}) \in M_n(R)$ ($1 \leq s, t \leq n$). Then there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in R$ such that $a_1^{(1)}x_1 + \cdots + a_1^{(n)}x_n + a_1^{(1)}y_1 + \cdots + a_1^{(n)}y_n = 1, \ldots, a_2^{(1)}x_1 + \cdots + a_2^{(n)}x_n + a_2^{(1)}y_1 + \cdots + a_2^{(n)}y_n = 0, \ldots, a_1^{(1)}x_1 + \cdots + a_1^{(n)}x_n + a_1^{(1)}y_1 + \cdots + a_1^{(n)}y_n = 0, \ldots, a_2^{(1)}x_1 + \cdots + a_2^{(n)}x_n + a_2^{(1)}y_1 + \cdots + a_2^{(n)}y_n = 0$. In view of Lemma 7, there is $z_1 \in R$ such that $a_1^{(1)} + a_1^{(2)}x_1z_1 + \cdots + a_1^{(n)}x_nz_1 + a_1^{(1)}y_1z_1 + \cdots + a_1^{(n)}y_nz_1 = u_1 \in U(R)$ and $1 - x_1z_1 = v_1 \in U_m(R)$. So we claim that
The sum of a that for any A is finite, then there exist submodules A_{ij} such that A = A' \oplus (\bigoplus_{i \in I} A_i), where M_k' \cong R_k and the index set I is finite, then there exist submodules A_i < A such that A = A' \oplus (\bigoplus_{i \in I} A_i). A ring R is said to be strongly \pi-regular provided that for any x \in R, there exists a positive integer n such that x^n = x^{n+1}y for some y \in R.

We note that R satisfies (m,1)-stable range if and only if it has stable range one and for any x, y \in R, there exists w \in U_m(R) such that x y + x w + 1 \in U(R). By an argument
of M. Henriksen [11], we claim that the ring $R$ has stable range one if and only if the ring $M_2(R)$ satisfies $(3,1)$-stable range. For exchange rings, we now derive the following.

**Lemma 10.** Let $R$ be an exchange ring with $1/2 \in R$. Then the following are equivalent:
1. The exchange ring $R$ has stable range one.
2. The exchange ring $R$ satisfies $(7,1)$-stable range.

**Proof.** (2)⇒(1). The proof is clear.

1. Whenever $aR + bR = dR$, there exist $y \in U_m(R)$, $u \in U(R)$ such that $a + by = du$.
2. Whenever $Ra + Rb = dR$, there exist $z \in U_m(R)$, $u \in U(R)$ such that $a + zb = ud$.

**Proposition 12.** The following are equivalent:
1. The ring $R$ satisfies $(m,1)$-stable range.
2. Whenever $aR + bR = dR$, there exist $y \in U_m(R)$, $u \in U(R)$ such that $a + by = du$.
3. Whenever $Ra + Rb = dR$, there exist $z \in U_m(R)$, $u \in U(R)$ such that $a + zb = ud$.

**Proof.** (1)⇒(2). Given $aR + bR = dR$, then $(a,b)M_2(R) = (d,0)M_2(R)$. Assume that $(d,0)A = (a,b)$ and $(a,b)B = (d,0)$. From $AB + (I_2 - AB) = I_2$, we have $Y \in M_2(R)$ such that $A + (I_2 - AB)Y = W \in GL_2(R)$. Thus $(a,b) = (d,0)A = (d,0)(A + (I_2 - AB)) = (d,0)W$. Assume that $W = (w_{ij})$. Then $w_{11}R + w_{12}R = R$, whence $w_{11} + w_{12}y = u \in U(R)$ for $y \in U_m(R)$. Therefore $a + by = du$, as desired.

(2)⇒(1). The proof is trivial.

(1)⇐(3). Applying (1)⇐(2) to the opposite ring $R^{op}$, we complete the proof by the symmetry of $(m,1)$-stable range property.

**Corollary 13.** Let $R$ be a ring which is quasi-injective as a right $R$-module. Then the following are equivalent:
1. The ring $R$ satisfies $(m,1)$-stable range.
2. Whenever $r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(d)$, there exists $z \in U_m(R)$ such that $r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(a + zb)$.
3. Whenever $l \cdot \ann(a) \cap l \cdot \ann(b) = l \cdot \ann(d)$, there exists $y \in U_m(R)$ such that $l \cdot \ann(a) \cap l \cdot \ann(b) = l \cdot \ann(a + by)$.

**Proof.** (1)⇒(2). Suppose $r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(d)$. By [5, Proposition 3.4], we claim that $Ra + Rb = Rd$. Using Proposition 12, we can find some $z \in U_m(R)$ such
that \( a + zb = du \) for some \( u \in U(R) \). Therefore \( r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(d) = r \cdot \ann(a + zb) \), as desired.

(2)\(\Rightarrow\)(1). Assume that \( Ra + Rb = R \). Then \( r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(1) \). Thus, we claim that \( r \cdot \ann(a) \cap r \cdot \ann(b) = r \cdot \ann(a + zb) \) for \( a \in U_m(R) \). Therefore \( r \cdot \ann(1) = r \cdot \ann(a + zb) \). By [5, Proposition 3.4], we show that \( R = R(a + zb) \), and then \( a + zb = u \) is left invertible in \( R \). Assume that \( vu = 1 \) for some \( v \in R \). From \( Rv + R(1 - uv) = R \), we also have \( w \in U_m(R) \) such that \( v + w(1 - uv) = t \) is left invertible in \( R \). Clearly, we have \( tu = (v + w(1 - uv))u = 1 \). Hence \( t \) is a unit of \( R \). Therefore \( a + zb = u \) is a unit of \( R \), as desired.

(1)\(\Leftrightarrow\)(2). By the symmetry of \((m,1)\)-stable range condition, we complete the proof.

\[ \square \]

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