CHARACTERIZATION OF AN $H^*$-ALGEBRAS IN TERMS OF A TRACE

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ABSTRACT. Arbitrary proper $H^*$-algebra is characterized in terms of the trace defined on a certain subset of the algebra.

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This paper deals with characterizations of an arbitrary proper (not necessarily commutative) $H^*$-algebra. More specifically, we show that any Banach $*$-algebra with a partially defined trace, and some additional properties, has a Hilbert space structure with respect to which it is an $H^*$-algebra. In the past, the author worked in characterizations of commutative $H^*$-algebras (e.g., [3, 4]). As in [3], we do not assume both the existence of an inner product and the commutativity.

Now, we state our first result.

**Theorem 1.** Let $A$ be a Banach algebra with an involution $x \to x^*$, $x \in A$, such that $\|x^*\| = \|x\|$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a complex valued trace, that is, there is a complex valued function $\text{tr}$ on $A^2$ with the following properties:

(i) if $x, y, and x + y$ belong to $A^2$, then $\text{tr}(x + y) = \text{tr}x + \text{tr}y$,

(ii) $\text{tr}(\lambda x) = \lambda \text{tr}x$ for all $x \in A^2$ and any complex number $\lambda$,

(iii) $\text{tr}(x^*x) \geq 0$ and $\text{tr}x^*x = 0$ if and only if $x = 0$, $x \in A$,

(iv) $\text{tr}x^* = \overline{\text{tr}x}$ for all $x \in A^2$.

Suppose also that

(v) $\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in A$,

(vi) $|\text{tr}(xy)| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$,

(vii) for each bounded linear functional $f(f \in A^*)$, there exists $a \in A$ such that $f(x) = \text{tr}(xa^*)$ for all $x \in A$.

Then $A$ is a proper $H^*$-algebra with respect to some scalar product whose corresponding norm $\|\|$ is equivalent to the original norm. This means that there exists a scalar product $(,)$ on $A$ such that $A$ is an $H^*$-algebra with respect to this scalar product (and the original involution) and such that $k_1\|x\|^2 \leq (x,x) \leq k_2\|x\|^2$ for some $k_1, k_2 \geq 0$ and all $x \in A$ (and $(xy,z) = (y,x^*z) = (x,zy^*)$ for all $x, y, z \in A$).

**Remark 2.** Note that each proper $H^*$-algebra $A$ has all the properties stated in Theorem 1 [5].

**Proof of Theorem 1.** For any $x, y \in A$, let $(x,y) = \text{tr}(xy^*) = \text{tr}(y^*x)$. Then $(,)$ is an inner product on $A$ [6] (in the terminology of Loomis [1], $(,)$ is a scalar product). Let $\|\|_2$ be the corresponding norm, $(x,x) = \|x\|_2^2$, $x \in A$. 
We show that $A$ is complete with respect to this new norm $\| \cdot \|_2$. Let $\{a_n\}$ be a Cauchy sequence, $\lim_{m,n} \|a_n - a_m\|_2 = 0$. Then there exists $M > 0$ such that $\|a_n\|_2 \leq M$ for all $n$ (every Cauchy sequence is bounded). For any fixed $x \in A$, the sequence $\{\text{tr}(xa_n^*)\}$ of complex numbers is also Cauchy

$$\left| \text{tr}(xa_n^*) - \text{tr}(xa_m^*) \right| \leq \|x\|_2 \|a_n - a_m\|_2. \quad (1)$$

So there is a complex number $\lambda_x$ such that $\text{tr}(xa_n^*) \rightarrow \lambda_x$, $n \rightarrow \infty$. Define the complex valued function $f$ on $A$ by setting $f(x) = \lambda_x$. It follows from

$$f(x) = \lim \text{tr}(xa_n^*), \quad \|a_n\|_2 \leq M, \quad (2)$$

and the linearity of $\text{tr}$ that $f$ is a bounded linear functional on $A(f \in A^*)$. Assumption (vii) in Theorem 1 implies that there exists $a \in A$ such that

$$(x,a) = f(x) = \lim \text{tr} xa_n^*. \quad (3)$$

We show that $\lim_{n \rightarrow \infty} \|a_n - a\|_2 = 0$. Let $\epsilon > 0$ be arbitrary, and let $n_0$ be such that $\|a_n - a_m\|_2 < \epsilon/2$ for all $n,m > n_0$. Let $n > n_0$ be fixed and arbitrary. The following relation:

$$\|a - a_n\|_2^2 = \left| (a - a_n, a - a_m) + (a - a_n, a_m - a_n) \right|$$
$$= \left| (a - a_n, a) - (a - a_n, a_m) + (a - a_n, a_m - a_n) \right|$$
$$\leq \|f(a - a_n) - (a - a_n, a_m)\| + \|a - a_n\|_2 \cdot \|a_n - a_m\|_2$$
$$\leq \|f(a - a_n) - (a - a_n, a_m)\| + \frac{\epsilon}{2} \|a - a_n\|_2$$

shows that $\|a - a_n\|_2^2 \leq \epsilon \|a - a_n\|_2$, since we can always find $m > n_0$ so that $\|f(a - a_n) - (a - a_n, a_m)\| \leq \epsilon/2 \|a - a_n\|_2$. Hence, $\|a - a_n\|_2 \leq \epsilon$ for any $n > n_0$. This proves that $A$ is complete in this new norm $\| \cdot \|_2$.

It follows from (vi) that $\|x\|_2 \leq \|x\| \cdot (x,x) \leq \|x\| \cdot \|x\|^* = \|x\|^2$ for all $x \in A$. Closed graph theorem [1] tells us that $\| \cdot \|_2$ is equivalent to the original norm.

Now, it is an easy exercise to show that $A$ is an $H^*$-algebra with respect to the inner product $(,)$.

Now we state a slightly different characterization. It may not look like much of improvement over Theorem 1, but it allows for a larger class of examples. In fact, if we take any proper $H^*$-algebra $A$ and replace its norm by any other norm equivalent to the original one, we get a canonical example of a Banach algebra which both satisfies the conditions of the following theorem and is characterized by it.

**Theorem 3.** Let $A$ be a Banach algebra with continuous involution $x \rightarrow x^*$, $x \in A$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a trace $\text{tr}$ with the following properties:

(i) if $x, y, x + y \in A^2$, then $\text{tr}(x + y) = \text{tr}x + \text{tr}y$,
(ii) $\text{tr}(\lambda x) = \lambda \text{tr}x$ for all $x \in A^2$ and each complex number $\lambda$,
(iii) $\text{tr}(x^*x) \geq 0$ and $\text{tr}(x^*x) = 0$ if and only if $x = 0$ $(x \in A)$,
(iv) $\text{tr}x^* = \overline{\text{tr}x}$, $x \in A^2$,
(v) $\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in A$. 


Assume further that
(vi)’ for each $a \in A$ the map $T_a : x \rightarrow \text{tr}(xa^*)(= T_a(x))$ is continuous ($T_a \in A^*$
for each $a \in A$),
(vii) for each bounded linear functional $f(f \in A^*)$ there exists $a \in A$ such that
$f(x) = \text{tr}(xa^*)$ for all $x \in A$.
Then $A$ has a structure of a proper $H^*$-algebra with respect to some scalar product $(,)$
such that $k \|x\|^2 \leq (x,x) \leq K \|x\|^2$ for all $x \in A$ and some $k, K > 0$.

**Remark 4.** Note that (vi)’ is equivalent to the following condition:
(vi)” there exists $M > 0$ such that $|\text{tr}(xy)| \leq M \|x\| \cdot \|y\|$ for all $x, y \in A$.

It is a consequence of uniform boundness theorem [6, page 239]. Proof of this fact is similar to
the proof of Lemma 1 in [2]. Note also that continuity of involution implies
that there exists $B > 0$ such that $\|x^*\| \leq B \|x\|$, $x \in A$.

**Proof of Theorem 3.** We leave it to the reader to modify the proof of Theorem 1
in order to verify validity of Theorem 3.

**References**


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