ON THE SOLVABILITY OF A VARIATIONAL INEQUALITY PROBLEM AND APPLICATION TO A PROBLEM OF TWO MEMBRANES

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(Received 17 March 2000)

ABSTRACT. The purpose of this work is to give a continuous convex function, for which we can characterize the subdifferential, in order to reformulate a variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$, $\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0$ as a system of independent equations, where $f$ belongs to $L^2(\Omega) \times L^2(\Omega)$ and $K = \{ v \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega \}$.

2000 Mathematics Subject Classification. Primary 35J85.

1. Introduction. We are interested in the following variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0,$$

where $f$ belongs to $L^2(\Omega) \times L^2(\Omega)$ and $K$ is a closed convex set of $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$K = \{ v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega \}. \tag{1.2}$$

Thanks to the orthogonal projection of the space $L^2(\Omega) \times L^2(\Omega)$ onto the cone $\mathcal{K}$ defined by

$$\mathcal{K} = \{ v = (v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega \}, \tag{1.3}$$

we construct a functional $\varphi$ for which we can characterize the subdifferential at a point $u$, in order to reformulate problem (1.1) to a variational inequality without constraints; that is, find $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + \varphi(v) - \varphi(u) + (h, v - u) \geq 0,$$

where $\varphi$ is a continuous convex function from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $\mathbb{R}$ and $h$ is an element of $L^2(\Omega) \times L^2(\Omega)$ depending only on $f$.

We prove that the solution $u = (u_1, u_2)$ can be obtained as a solution of a system of independent two Dirichlet problems

$$u_1, u_2 \in H_0^1(\Omega), \quad \Delta u_1 = g_1, \quad \Delta u_2 = g_2 \text{ in } \Omega, \tag{1.5}$$

where $g_1$ and $g_2$ are two functions of $L^2(\Omega)$ determined in terms of $f_1$ and $f_2$. We will give an algorithm for computing these functions.
This approach can be applied to study a variational inequality arising from a problem of two membranes [2].

2. Formulation of the problem. Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. We equip $H^1_0(\Omega) \times H^1_0(\Omega)$ with the norm

$$a(u, v) = \int_\Omega \nabla u_1 \nabla v_1 + \int_\Omega \nabla u_2 \nabla v_2,$$  \tag{2.1}$$

where

$$u = (u_1, u_2), \quad v = (v_1, v_2) \in H^1_0(\Omega) \times H^1_0(\Omega).$$  \tag{2.2}$$

For $r \in L^2(\Omega)$, we let

$$r^+ = \max\{r, 0\}, \quad r^- = \min\{r, 0\}. \tag{2.3}$$

For $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$, we let

$$f^+ = (f_1^+, f_2^+), \quad f^- = (f_1^-, f_2^+). \tag{2.4}$$

For $v = (v_1, v_2) \in H^1_0(\Omega) \times H^1_0(\Omega)$, we let

$$v_+ = \left( v_1 + \frac{(v_2 - v_1)^+}{2}, v_2 - \frac{(v_2 - v_1)^+}{2} \right), \quad v_- = \left( - \frac{(v_2 - v_1)^+}{2}, \frac{(v_2 - v_1)^+}{2} \right) \tag{2.5}$$

the projection of $v$ onto the cone $\mathcal{K}$ given by (1.3) with respect to the scalar product of $L^2(\Omega) \times L^2(\Omega)$ (respectively, the projection with respect to the scalar product of $L^2(\Omega) \times L^2(\Omega)$ on the polar cone of $\mathcal{K}$ defined by $\mathcal{K}^0 = \{ v = (-r, r) \in L^2(\Omega) \times L^2(\Omega) : r \geq 0 \text{ a.e. on } \Omega \}$). We easily verify that

$$a(v_+, v_-) = 0 \tag{2.6}$$

for all $v \in H^1_0(\Omega) \times H^1_0(\Omega)$. A function $\varphi$ defined from $H^1_0(\Omega) \times H^1_0(\Omega)$ to $\mathbb{R}$ is called lower semi-continuous (l.s.c.) if its epigraph defined by

$$\text{epi(} \varphi \text{)} = \{ v = (v_1, v_2) \in H^1_0(\Omega) \times H^1_0(\Omega), \lambda \in \mathbb{R} : \varphi(v) \leq \lambda \} \tag{2.7}$$

is closed in $H^1_0(\Omega) \times H^1_0(\Omega) \times \mathbb{R}$. Let $u \in H^1_0(\Omega) \times H^1_0(\Omega)$, we denote by $\partial \varphi(u)$ the subdifferential of $\varphi$ at $u$, defined by

$$\partial \varphi(u) = \{ \mu \in H^{-1}(\Omega) \times H^{-1}(\Omega) : \varphi(u) - \varphi(v) \leq \langle \mu, u - v \rangle \forall v \in H^1_0(\Omega) \times H^1_0(\Omega) \}. \tag{2.8}$$

If $\varphi$ is a convex l.s.c. function, then for all $v \in H^1_0(\Omega) \times H^1_0(\Omega)$, $\partial \varphi(v) \neq \emptyset$.

Let $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$. We denote by $(\cdot, \cdot)$ and $\| \cdot \|$ the scalar product and the norm of $L^2(\Omega) \times L^2(\Omega)$, respectively. We consider the following variational inequality problem: find $u = (u_1, u_2) \in K$ such that

$$a(u, v - u) + (f, v - u) \geq 0 \quad \forall v = (v_1, v_2) \in K. \tag{2.9}$$

It admits a unique solution. The functional $\varphi$ defined from $L^2(\Omega) \times L^2(\Omega)$ to $\mathbb{R}$ by

$$v \rightarrow (f^+, v_+)$$

is continuous on $H^1_0(\Omega) \times H^1_0(\Omega)$ and convex.
**Proposition 2.1.** \( u = (u_1, u_2) \) is a solution of the problem (2.9) if and only if \( u \) is the solution of the following problem: find \( u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega) \) such that

\[
ad(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega).
\]

**Proof.** It is well known in the general theory of variational inequalities that problem (2.10) admits a unique solution. So, it is sufficient to show that the solution \( u \) of (2.10) is an element of \( K \). Let \( v = u_+ \), then the inequality of (2.10) becomes

\[
ad(u, -u_+) + \varphi(u) - \varphi(u) + (f^-, -u_+) \geq 0.
\]

By the relation (2.6) we deduce that \( u_+ = 0 \), hence \( u \in K \). \( \square \)

**Proposition 2.2.** Problem (2.10) is equivalent to the following problem: find \( u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega) \), \( \varphi(u) \in H_0^1(\Omega) \times H_0^1(\Omega) \),

\[
ad(u, v) + (\varphi(u), v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \quad \varphi(u) \in \partial \varphi(u).
\]

**Proof.** If \( u \in H_0^1(\Omega) \times H_0^1(\Omega) \) and \( \mu \in L^2(\Omega) \times L^2(\Omega) \) are the solution of (2.12), then by definition of \( \mu \in \partial \varphi(u) \), we have

\[
ad(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega).
\]

Conversely, let \( u \) be the solution of problem (2.10). For \( v = u \pm w \), with \( w \in H_0^1(\Omega) \times H_0^1(\Omega) \), the inequality of (2.10) gives

\[
\begin{align*}
ad(u, w) + (f^-, w) & \geq -(f^+, w^+) \geq -\|f^+\| \|w\|, \\
ad(u, w) + (f^-, w) & \leq (f^+, (-w)^+) \leq \|f^+\| \|w\|.
\end{align*}
\]

We deduce that

\[
|a(u, w) + (f^-, w)| \leq \|f^+\| \|w\|.
\]

So the linear form

\[
w \mapsto a(u, w) + (f^-, w)
\]

is continuous on \( H_0^1(\Omega) \times H_0^1(\Omega) \) equipped with the norm of \( L^2(\Omega) \times L^2(\Omega) \). Where \( \mu \) is an element of \( L^2(\Omega) \times L^2(\Omega) \). \( \square \)

We set

\[
C = \{ v \in L^2(\Omega) \times L^2(\Omega), \ (v, v) \leq \varphi(v) \ \forall v \in L^2(\Omega) \times L^2(\Omega) \}.
\]

**Lemma 2.3.** Let \( u \in L^2(\Omega) \times L^2(\Omega) \), then the following properties are equivalent:

(a) \( \mu \in \partial \varphi(u) \).

(b) \( \mu \in C \) and \( (\mu, u) = \varphi(u) \).

(c) \( \mu \in C \) and \( (v - \mu, u) \leq 0 \) for all \( v \in C \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( \mu \in \partial \varphi(u) \), we have

\[
\varphi(v) - \varphi(u) \geq (\mu, v - u) \quad \forall v \in L^2(\Omega) \times L^2(\Omega).
\]

\[
C = \{ v \in L^2(\Omega) \times L^2(\Omega), \ (v, v) \leq \varphi(v) \ \forall v \in L^2(\Omega) \times L^2(\Omega) \}. 
\]
We put $v = 0$, next $v = 2u$ in (2.18). Since $\varphi$ is positively homogeneous of degree 1, we obtain $\varphi(u) = (\mu, u)$ and consequently
\[ \varphi(v) \geq (\mu, v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \] (2.19)

(c)$\Rightarrow$(a). For all $v \in V$, we have
\[ (\mu, v - u) \leq \varphi(v) - (\mu, u) \leq \varphi(v) - (v, u) \quad \forall v \in C. \] (2.20)
Hence for $v \in \partial \varphi(u)$, we have $(\nu, u) = \varphi(u)$, consequently $\mu \in \varphi(u)$. \hfill \Box

We deduce from Lemma 2.3 the following relations:
\[ \mu_1 + \mu_2 = f_1^+ + f_2^-, \quad f_2^- \leq \mu_2 \leq \mu_1 \leq f_1^+ \text{ a.e. in } \Omega. \] (2.21)

Indeed, the function $\varphi$ being positively homogeneous of degree 1, $\mu \in \partial \varphi(u)$ implies
\[ (\mu, u) = \varphi(u), \] (2.22)
\[ (\mu, v) \leq \varphi(v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \] (2.23)
Finally, it is sufficient to take in (2.23) elements $v = (v_1, v_2)$ with suitable choices on the components $v_1$ and $v_2$.

Let $V = H_0^1(\Omega) \times H_0^1(\Omega)$, and taking into account Lemma 2.3, we can write problem (2.12) as follows: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $\mu \in C$,
\[ a(u, v) + (\mu, v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \] (2.24)
\[ (v - \mu, u) \leq 0 \quad \forall v \in C. \]
Let $A$ be the Riesz-Fréchet representation of $H^{-1}(\Omega) \times H^{-1}(\Omega)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. We set $M = A(C)$, this is a closed convex subset in $H_0^1(\Omega) \times H_0^1(\Omega)$ characterized by
\[ M = \{ w \in H_0^1(\Omega) \times H_0^1(\Omega) : a(w, v) \leq \varphi(v) \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega) \}. \] (2.25)
Problem (2.24) can be written in the following form: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $z \in M$,
\[ a(u + z + t, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \] (2.26)
\[ a(w - z, u) \leq 0 \quad \forall w \in M. \]
with $z = A(\mu)$ and $t = A(f^-)$. Hence
\[ u = -z - t, \quad z = P_M(-t), \] (2.27)
where $P_M(-t)$ is the projection of $-t$ onto the closed convex set $M$ with respect to the scalar product $a(\cdot, \cdot)$ of $H_0^1(\Omega) \times H_0^1(\Omega)$.

From the equality of Proposition 2.2, we deduce that the solution $u$ of problem (2.9) verifies the following equations:
\[ \Delta u_1 = \mu_1 + f_1^-, \quad \Delta u_2 = \mu_2 + f_2^+ \text{ in } \Omega. \] (2.28)
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We notice that the prior knowledge of \( \mu = (\mu_1, \mu_2) \) in terms of data of problem (2.9) yields the solutions \( u_1 \) and \( u_2 \) as solutions of two independent Dirichlet problems given by the system (2.28). We recall that for each element \( f \) of \( L^p(\Omega) \), the solution of the problem

\[
u \in H^1_0(\Omega), \quad -\Delta u = f \quad \text{in } \Omega,
\]

verifies the following properties (see [2]):

\[
u \in H^{2,p}(\Omega), \quad \| \nu \|_{H^{2,p}} \leq C \| f \|_{L^p},
\]

where \( C \) is a constant depending only on \( p \) and \( \Omega \). We deduce from (2.28) that \( u_1, u_2 \) are in \( H^2(\Omega) \) and

\[
\| u_1 \|_{H^2(\Omega)} \leq c_1 \| \mu_1 + f^+_1 \|_{L^2(\Omega)},
\]

\[
\| u_2 \|_{H^2(\Omega)} \leq c_2 \| \mu_2 + f^-_2 \|_{L^2(\Omega)},
\]

\[
\| u_1 + u_2 \|_{H^2(\Omega)} \leq c \| f_1 + f_2 \|_{L^2(\Omega)},
\]

where \( c, c_1, \) and \( c_2 \) are constants depending only on \( \Omega \). We define the domain of non-coincidence [2] by

\[
\Omega^+ = \{ x \in \Omega : u_1(x) > u_2(x) \}.
\]

From relations (2.21), (2.22), and (2.23) we deduce that

\[
\mu_1 = f^+_1, \quad \mu_2 = f^-_2 \quad \text{a.e. in } \Omega^+.
\]

When \( u_1 \) and \( u_2 \) are continuous on \( \Omega \), the following relations are verified:

\[
\Delta u_1 = f_1, \quad \Delta u_2 = f_2 \quad \text{in } \Omega^+.
\]

2.1. Algorithm for computing \( \nu \). We consider the following projection problem:

\[
z \in H^1_0(\Omega) \times H^1_0(\Omega), \quad z = P_M(t'), \quad \text{where } t' = -t.
\]

Let \( z_0 \) belong to \( M \), we compute the element \( w_0 \) of \( M \) which verifies the following inequality:

\[
a(w - w_0, z_0 - t') \geq 0 \quad \forall w \in M.
\]

Next we compute

\[
z_1 = P_{\{z_0, w_0\}}(t').
\]

So, the algorithm is: \( z_n \) being given in \( M \), we construct \( w_n \) verifying

\[
a(w - w_n, z_n - t') \geq 0 \quad \forall w \in M.
\]

Next \( z_{n+1} = P_{\{z_n, w_n\}}(t') \). The sequence \( \{z_n\} \) converges in \( H^1_0(\Omega) \times H^1_0(\Omega) \) strongly to the solution of problem (2.35) [1]. Since \( M = A(C) \), then the inequality (2.38) implies that there exists \( \{v_n\} \) in \( C \) which verifies

\[
(v - v_n, t' - z_n) \leq 0 \quad \forall v \in C
\]

and Lemma 2.3 shows that \( v_n \) is an element of \( \partial \varphi(t' - z_n) \).
2.2. Application. This method of solvability can be applied to the study of a variational inequality arising from a problem of two membranes [2],
\[
\begin{align*}
\Delta u_1 + \lambda u_1 &= f_1, \quad \Delta u_2 = f_2 \text{ in } \Omega^+, \quad u_1 = u_2, \\
\frac{\partial u_1}{\partial x_i} &= \frac{\partial u_2}{\partial x_i}, \quad 1 \leq i \leq n, \\
\Delta u_1 + \left(\frac{\lambda}{2}\right) u_1 &= \frac{1}{2} (f_1 + f_2) \text{ in } \Omega^-,
\end{align*}
\]
(2.40)
where \(\Omega^+\) and \(\Omega^-\), are two parts of \(\Omega\) (unknown) separated by a hypersurface \(\Gamma\) of \(\mathbb{R}^n\) such that \(\Omega = \Omega^+ \cup \Gamma \cup \Omega^-\); \(f_1, f_2\) are two regular functions and \(\lambda \in \mathbb{R}\). Formally, \(\Omega^+\) is the non-coincidence domain given by (2.32).

References
