A NOTE ON $\theta$-GENERALIZED CLOSED SETS

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ABSTRACT. The purpose of this note is to strengthen several results in the literature concerning the preservation of $\theta$-generalized closed sets. Also conditions are established under which images and inverse images of arbitrary sets are $\theta$-generalized closed. In this process several new weak forms of continuous functions and closed functions are developed.

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1. Introduction. Recently Dontchev and Maki [5] have introduced the concept of a $\theta$-generalized closed set. This class of sets has been investigated also by Arockiarani et al. [1]. The purpose of this note is to strengthen slightly some of the results in [5] concerning the preservation of $\theta$-generalized closed sets. This is done by using the notion of a $\theta$-c-closed set developed by Baker [2]. These sets turn out to be a very natural tool to use in investigating the preservation of $\theta$-generalized closed sets. In this process we introduce a new weak form of a continuous function and a new weak form of a closed function, called $\theta$-$g$-$c$-continuous and $\theta$-$g$-$c$-closed, respectively. It is shown that $\theta$-$g$-$c$-continuity is strictly weaker than strong $\theta$-continuity and that $\theta$-$g$-$c$-closed is strictly weaker than $\theta$-$g$-closed.

2. Preliminaries. The symbols $X$ and $Y$ denote topological spaces with no separation axioms assumed unless explicitly stated. If $A$ is a subset of a space $X$, then the closure and interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. The $\theta$-closure of $A$ [8], denoted by $\text{Cl}_\theta(A)$, is the set of all $x \in X$ for which every closed neighborhood of $x$ intersects $A$ nontrivially. A set $A$ is called $\theta$-closed if $A = \text{Cl}_\theta(A)$. The $\theta$-interior of $A$ [8], denoted by $\text{Int}_\theta(A)$, is the set of all $x \in X$ for which $A$ contains a closed neighborhood of $x$. A set $A$ is said to be $\theta$-open provided that $A = \text{Int}_\theta(A)$. Furthermore, the complement of a $\theta$-open set is $\theta$-closed and the complement of a $\theta$-closed set is $\theta$-open.

Definition 2.1 (Dontchev and Maki [5]). A set $A$ is said to be $\theta$-generalized closed (or briefly $\theta$-$g$-closed) provided that $\text{Cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open. A set is called $\theta$-generalized open (or briefly $\theta$-$g$-open) if its complement is $\theta$-generalized closed.

The following theorem from [5] gives a useful characterization of $\theta$-$g$-openness.
Theorem 2.2 (Dontchev and Maki [5]). A set $A$ is $\theta$-g-open if and only if $F \subseteq \text{Int}_\theta(A)$ whenever $F \subseteq A$ and $F$ is closed.

Definition 2.3 (Dontchev and Maki [5]). A function $f : X \to Y$ is said to be $\theta$-g-closed provided that $f(A)$ is $\theta$-g-closed in $Y$ for every closed subset $F$ of $X$.

Definition 2.4 (Dontchev and Maki [5]). A function $f : X \to Y$ is said to be $\theta$-g-irresolute ($\theta$-g-continuous), if for every $\theta$-g-closed (closed) subset $A$ of $Y$, $f^{-1}(A)$ is $\theta$-$g$-closed in $X$.

Definition 2.5 (Noiri [7]). A function $f : X \to Y$ is said to be strongly $\theta$-continuous provided that, for every $x \in X$ and every open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ for which $f(\text{Cl}(U)) \subseteq V$.

3. Sufficient conditions for images of $\theta$-g-closed sets to be $\theta$-g-closed. Dontchev and Maki [5] proved that the $\theta$-g-closed, continuous image of a $\theta$-g-closed set is $\theta$-g-closed. In this section, we strengthen this result by replacing both the $\theta$-g-closed and continuous requirements with weaker conditions. Our replacement for the $\theta$-g-closed condition uses the concept of a $\theta$-c-open set from [2].

Definition 3.1 (Baker [2]). A set $A$ is said to be $\theta$-c-closed provided there is a set $B$ for which $A = \text{Cl}_\theta(B)$.

We define a function $f : X \to Y$ to be $\theta$-g-closed if $f(A)$ is $\theta$-g-closed in $Y$ for every $\theta$-c-closed set $A$ in $X$. Since $\theta$-c-closed sets are obviously closed, $\theta$-g-closed implies $\theta$-g-c-closed. The following example shows that the converse implication does not hold.

Example 3.2. Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and let $f : X \to X$ be the identity mapping. Since the $\theta$-closure of every nonempty set is $X$, $f$ is obviously $\theta$-g-c-closed. However, since $f(\{c\})$ fails to be $\theta$-g-closed, $f$ is not $\theta$-g-closed.

Theorem 3.3. If $f : X \to Y$ is continuous and $\theta$-g-c-closed, then $f(A)$ is $\theta$-g-closed in $Y$ for every $\theta$-g-closed set $A$ in $X$.

Proof. Assume $A$ is a $\theta$-g-closed subset of $X$ and that $f(A) \subseteq V$, where $V$ is an open subset of $Y$. Then $A \subseteq f^{-1}(V)$, which is open. Since $A$ is $\theta$-g-closed, $\text{Cl}_\theta(A) \subseteq f^{-1}(V)$ and hence $f(\text{Cl}_\theta(A)) \subseteq V$. Because $\text{Cl}_\theta(A)$ is $\theta$-c-closed and $f$ is $\theta$-g-c-closed, $f(\text{Cl}_\theta(A))$ is $\theta$-g-closed. Therefore $\text{Cl}_\theta(f(\text{Cl}_\theta(A))) \subseteq V$ and hence $\text{Cl}_\theta(f(A)) \subseteq \text{Cl}_\theta(f(\text{Cl}_\theta(A))) \subseteq V$, which proves that $f(A)$ is $\theta$-g-closed.

Corollary 3.4 (Dontchev and Maki [5]). If $f : X \to Y$ is continuous and $\theta$-g-closed, then $f(A)$ is $\theta$-g-closed in $Y$ for every $\theta$-g-closed subset $A$ of $X$.

Theorem 3.3 can be strengthened further by replacing continuity with a weaker condition. Instead of requiring inverse images of open sets to be open, we require that the inverse images of open sets interact with $\theta$-g-closed sets in the same way as open sets.
**Definition 3.5.** A function \( f : X \to Y \) is said to be approximately \( \theta \)-continuous (or briefly \( a-\theta \)-continuous) provided that \( \text{Cl}_\theta(A) \subseteq f^{-1}(V) \) whenever \( A \subseteq f^{-1}(V) \), \( A \) is \( \theta-g \)-closed, and \( V \) is open.

The proof of Theorem 3.3 is easily modified to obtain the following result.

**Theorem 3.6.** If \( f : X \to Y \) is \( a-\theta \)-continuous and \( \theta-g \)-closed, then \( f(A) \) is \( \theta-g \)-closed in \( Y \) for every \( \theta-g \)-closed set \( A \) in \( X \).

Obviously continuity implies \( a-\theta \)-continuity and the following example shows that \( a-\theta \)-continuity is strictly weaker than continuity.

**Example 3.7.** Let \((X, \tau)\) be the space in Example 3.2 and let \( \sigma = \{X, \emptyset, \{b\}\} \). Then the identity mapping \( f : (X, \tau) \to (X, \sigma) \) is not continuous but is \( a-\theta \)-continuous.

In [4] Dontchev defined a function to be contra-continuous provided that inverse images of open sets are closed. We modify this concept slightly to obtain a \( \theta \)-contra-continuous function.

**Definition 3.8.** A function \( f : X \to Y \) is said to be \( \theta \)-contra-continuous if for every open subset \( V \) of \( Y \), \( f^{-1}(V) \) is \( \theta \)-closed.

If the continuity requirement in Theorem 3.3 is replaced with \( \theta \)-contra-continuity, then a much stronger result is obtained. The step in the proof of Theorem 3.3 where we obtain \( \text{Cl}_\theta(A) \subseteq f^{-1}(V) \) now holds for every set \( A \), because \( f^{-1}(V) \) is \( \theta \)-closed. Therefore we have the following theorem.

**Theorem 3.9.** If \( f : X \to Y \) is \( \theta \)-contra-continuous and \( \theta-g \)-closed, then \( f(A) \) is \( \theta-g \)-closed in \( Y \) for every subset \( A \) of \( X \).

4. **Sufficient conditions for \( \theta-g \)-irresoluteness.** Dontchev and Maki[5] proved that a strongly \( \theta \)-continuous, closed function is \( \theta-g \)-irresolute. We strengthen this result slightly by replacing strong \( \theta \)-continuity and closure with weaker conditions.

We define a function \( f : X \to Y \) to be \( \theta-g \)-continuous provided that, for every \( \theta-g \)-closed subset \( A \) of \( Y \), \( f^{-1}(A) \) is \( \theta-g \)-closed. Since strong \( \theta \)-continuity is equivalent to the requirement that inverse images of closed sets be \( \theta-g \)-closed [6], strong \( \theta \)-continuity obviously implies \( \theta-g \)-continuity. The function in Example 3.2 is \( \theta-g \)-continuous but not strongly \( \theta \)-continuous.

**Theorem 4.1.** If \( f : X \to Y \) is \( \theta-g \)-continuous and closed, then \( f \) is \( \theta-g \)-irresolute.

**Proof.** Assume \( A \subseteq Y \) is \( \theta-g \)-closed and that \( f^{-1}(A) \subseteq U \), where \( U \) is open. Then \( X - U \subseteq X - f^{-1}(A) \) and we see that \( f(X - U) \subseteq Y - A \). Since \( A \) is \( \theta-g \)-closed, \( Y - A \) is \( \theta-g \)-open. Also, since \( f \) is closed, \( f(X - U) \) is closed. Thus \( f(X - U) \subseteq \text{Int}_\theta(Y - A) = Y - \text{Cl}_\theta(A) \) or \( X - U \subseteq f^{-1}(Y - \text{Cl}_\theta(A)) = X - f^{-1}(\text{Cl}_\theta(A)) \) and we have that \( f^{-1}((\text{Cl}_\theta(A)) \subseteq U \). Since \( f \) is \( \theta-g \)-continuous, \( f^{-1}(\text{Cl}_\theta(A)) \) is \( \theta-g \)-closed. Therefore \( \text{Cl}_\theta(f^{-1}(A)) \subseteq \text{Cl}_\theta(f^{-1}(\text{Cl}_\theta(A))) \subseteq U \), which proves that \( f^{-1}(A) \) is \( \theta-g \)-closed. Thus \( f \) is \( \theta-g \)-irresolute. \( \Box \)
Corollary 4.2 (Dontchev and Maki [5]). If \( f : X \to Y \) is strongly \( \theta \)-continuous and closed, then \( f \) is \( \theta \)-\( g \)-irresolute.

Obviously \( \theta \)-\( g \)-continuity implies \( \theta \)-\( g \)-\( c \)-continuity. Therefore we have the following result.

Corollary 4.3. If \( f : X \to Y \) is \( \theta \)-\( g \)-continuous and closed, then \( f \) is \( \theta \)-\( g \)-irresolute.

The function in Example 3.2 is \( \theta \)-\( g \)-\( c \)-continuous but not \( \theta \)-\( g \)-continuous.

Theorem 4.1 can be strengthened in much the same way as Theorem 3.3 was strengthened by replacing the closure requirement with a weaker condition.

Definition 4.4. A function \( f : X \to Y \) is said to be approximately \( \theta \)-closed (or briefly \( a \)-\( \theta \)-closed) provided that \( f(F) \subseteq \text{Int}_{\theta}(A) \) whenever \( f(F) \subseteq A \), \( F \) is closed, and \( A \) is \( \theta \)-\( g \)-open.

Note that, under an \( a \)-\( \theta \)-closed function, images of closed sets interact with \( \theta \)-\( g \)-open sets in the same manner as closed sets. Obviously closed functions are \( a \)-\( \theta \)-closed. The inverse of the function in Example 3.7 is \( a \)-\( \theta \)-closed but not closed. The proof of the following theorem is analogous to that of Theorem 4.1.

Theorem 4.5. If \( f : X \to Y \) is \( \theta \)-\( g \)-\( c \)-continuous and \( a \)-\( \theta \)-closed, then \( f \) is \( \theta \)-\( g \)-irresolute.

Finally, Theorem 4.1 can be modified by replacing the requirement that the function be closed with a variation of a contra-closed function. Contra-closed functions, introduced by Baker [3], are characterized by having open images of closed sets.

Definition 4.6. A function \( f : X \to Y \) is said to be \( \theta \)-contra-closed provided that \( f(F) \) is \( \theta \)-open for every closed subset \( F \) of \( X \).

Theorem 4.7. If \( f : X \to Y \) is \( \theta \)-\( g \)-\( c \)-continuous and \( \theta \)-contra-closed, then for every subset \( A \) of \( Y \) \( f^{-1}(A) \) is \( \theta \)-\( g \)-closed (and hence also \( \theta \)-\( g \)-open).

The proof of Theorem 4.7 follows that of Theorem 4.1, except that the step \( f(X - U) \subseteq \text{Int}_{\theta}(Y - A) \) holds for every subset \( A \) of \( Y \) because \( f(X - U) \) is \( \theta \)-open.

References


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