ABOUT INTERPOLATION OF SUBSPACES OF REARRANGEMENT INVARIANT SPACES GENERATED BY RADEMACHER SYSTEM

SERGEY V. ASTASHKIN

(Received 1 August 2000 and in revised form 27 November 2000)

ABSTRACT. The Rademacher series in rearrangement invariant function spaces “close” to the space $L_\infty$ are considered. In terms of interpolation theory of operators, a correspondence between such spaces and spaces of coefficients generated by them is stated. It is proved that this correspondence is one-to-one. Some examples and applications are presented.

2000 Mathematics Subject Classification. Primary 46B70; Secondary 46B42, 42A55.

1. Introduction. Let

$$r_k(t) = \text{sign} \sin 2^{k-1} \pi t \quad (k = 1, 2, \ldots)$$

(1.1)

be the Rademacher functions on the segment $[0, 1]$. Define the linear operator

$$T a(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{for } a = (a_k)_{k=1}^{\infty} \in l_2.$$ (1.2)

It is well known (cf. [23, pages 340–342]) that $T a$ is an almost everywhere finite function on $[0, 1]$. Moreover, from Khintchine’s inequality it follows that

$$\|T a\|_{L_p} \asymp \|a\|_2 \quad \text{for } 1 \leq p < \infty,$$ (1.3)

where \(\|a\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}\). The symbol \(\asymp\) means the existence of two-sided estimates with constants depending only on \(p\). Also, it can easily be checked that

$$\|T a\|_{L_\infty} = \|a\|_1.$$ (1.4)

A more detailed information on the behaviour of Rademacher series can be obtained by treating them in the framework of general rearrangement invariant spaces.

Recall that a Banach space $X$ of measurable functions $x = x(t)$ on $[0, 1]$ is said to be a rearrangement invariant space (r.i.s.) if the inequality $x^*(t) \leq y^*(t)$, for $t \in [0, 1]$ and $y \in X$, implies $x \in X$ and $\|x\| \leq \|y\|$. Here and in what follows $z^*(t)$ is the nonincreasing rearrangement of a function $|z(t)|$ with respect to the Lebesgue measure denoted by $\text{meas}$ [10, page 83].

Important examples of r.i.s.’s are Marcinkiewicz and Orlicz spaces. Let $\mathcal{P}$ denote the cone of nonnegative increasing concave functions on the semiaxis $(0, \infty)$.

If $\varphi \in \mathcal{P}$, then the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x = x(t)$ such that

$$\|x\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_0^t x^*(s) \, ds : 0 < t \leq 1 \right\} < \infty.$$ (1.5)
If $S(t)$ is a nonnegative convex continuous function on $[0, \infty)$, $S(0) = 0$, then the Orlicz space $L_S$ consists of all measurable functions $x = x(t)$ such that
\[
\|x\|_S = \inf \left\{ u > 0 : \int_0^1 S\left(\frac{|x(t)|}{u}\right) dt \leq 1 \right\} < \infty.
\]

(1.6)

In particular, if $S(t) = t^p$ ($1 \leq p < \infty$), then $L_S = L^p$.

For any r.i.s. $X$ on $[0, 1]$ we have $L_\infty \subset X \subset L_1$ [10, page 124]. Let $X^0$ denote the closure of $L_\infty$ in an r.i.s. $X$.

In problems discussed below, a special role is played by the Orlicz space $L_N$, where $N(t) = \exp(t^2) - 1$ or, more precisely, by the space $G = L^0_N$. In [19], V. A. Rodin and E. M. Semenov proved a theorem about the equivalence of Rademacher system to the standard basis in the space $l_2$.

**Theorem 1.1.** Suppose that $X$ is an r.i.s. Then
\[
\|Ta\|_X = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp \|a\|_2
\]

(1.7)

if and only if $X \supset G$.

By Theorem 1.1, the space $G$ is the minimal space among r.i.s.’s $X$ such that the Rademacher system is equivalent in $X$ to the standard basis of $l_2$.

In this paper, we consider problems related to the behaviour of Rademacher series in r.i.s.’s intermediate between $L_\infty$ and $G$. Here a major role is played by concepts and methods of interpolation theory of operators.

For a Banach couple $(X_0, X_1)$, $x \in X_0 + X_1$ and $t > 0$, we introduce the Peetre $\mathcal{K}$-functional
\[
\mathcal{K}(t; x; X_0, X_1) = \inf \left\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \right\}.
\]

(1.8)

Let $Y_0$ be a subspace of $X_0$ and $Y_1$ a subspace of $X_1$. A couple $(Y_0, Y_1)$ is called a $\mathcal{K}$-subcouple of a couple $(X_0, X_1)$ if
\[
\mathcal{K}(t, y; Y_0, Y_1) \asymp \mathcal{K}(t, y; X_0, X_1),
\]

(1.9)

with constants independent of $y \in Y_0 + Y_1$ and $t > 0$.

In particular, if $Y_i = P(X_i)$, where $P$ is a linear projector bounded from $X_i$ into itself for $i = 0, 1$, then $(Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $(X_0, X_1)$ (see [3] or [21, page 136]). At the same time, there are many examples of subcouples that are not $\mathcal{K}$-subcouples (see [21, page 589], [22], and Remark 3.2 of this paper).

Let $T(l_1)$ (respectively $T(l_2)$) denote the subspace of $L_\infty$ (of $G$) consisting of all functions of the form $x = Ta$, where $T$ is given by (1.2) and $a \in l_1 (\in l_2)$. From (1.4) and Theorem 1.1 it follows that
\[
\mathcal{K}(t, Ta; T(l_1), T(l_2)) \asymp \mathcal{K}(t, a; l_1, l_2).
\]

(1.10)

In spite of the fact that $T(l_1)$ is uncomplemented in $L_\infty$ (see [17] or [11, page 134]) the following assertion holds.
**Theorem 1.2.** The couple \((T(l_1), T(l_2))\) is a \(\mathcal{H}\)-subcouple of the couple \((L_\infty, G)\). In other words (see \((1.10))\
\[ \mathcal{H}(t, T a; L_\infty, G) \cong \mathcal{H}(t, a; l_1, l_2), \tag{1.11} \]
with constants independent of \(a = (a_k)_{k=1}^\infty \in l_2\) and \(t > 0\).

We will use in the proof of **Theorem 1.2** an assertion about the distribution of Rademacher sums. It was proved by S. Montgomery-Smith [13].

**Theorem 1.3.** There exists a constant \(A \geq 1\) such that for all \(a = (a_k)_{k=1}^\infty \in l_2\) and \(t > 0\)
\[ \operatorname{meas} \left\{ s \in [0, 1] : \sum_{k=1}^\infty a_k r_k(s) > \varphi_a(t) \right\} \leq \exp \left( -\frac{t^2}{2} \right), \]
\[ \operatorname{meas} \left\{ s \in [0, 1] : \sum_{k=1}^\infty a_k r_k(s) > A^{-1} \varphi_a(t) \right\} \geq A^{-1} \exp \left( -A t^2 \right), \tag{1.12} \]
where \(\varphi_a(t) = \mathcal{H}(t, a; l_1, l_2)\).

Now we need some definitions from interpolation theory of operators. We say that a linear operator \(U\) is bounded from a Banach couple \((X, Y)\) into a Banach couple \((X_0, Y_0)\) (in short, \(U : X \rightarrow Y\)) if \(U\) is defined on \(X_0 + X_1\) and acts as bounded operator from \(X_i\) into \(Y_i\) for \(i = 0, 1\).

Let \(X = (X_0, X_1)\) be a Banach couple. A space \(X\) such that \(X_0 \cap X_1 \subset X \subset X_0 + X_1\) is called an interpolation space between \(X_0\) and \(X_1\) if each linear operator \(U : X \rightarrow X\) is bounded from \(X\) into itself.

To every r.i.s. \(X\) assign the sequence space \(FX\) of Rademacher coefficients of functions of the form \((1.2)\) from \(X\):
\[ \|(a_k)\|_{FX} = \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X. \tag{1.13} \]
Well-known properties of Rademacher functions imply that \(FX\) is an r.i. sequence space [19]. Furthermore, **Theorem 1.3** and properties of the \(\mathcal{H}\)-functional show that \(FX\) is an interpolation space between \(l_1\) and \(l_2\) (see the proof of **Theorem 1.2** later). For interpolation r.i.s. between \(L_\infty\) and \(G\) the correspondence \(X \rightarrow FX\) can be defined by using the real interpolation method.

For every \(p \in [1, \infty]\), we denote by \(l_p(u_k), u_k \geq 0 (k = 0, 1, \ldots)\) the space of all two-sided sequences of real numbers \(a = (a_k)_{k=-\infty}^{\infty}\) such that the norm \(\|a\|_{l_p(u_k)} = \|\{a_k u_k\}\|_p\) is finite. Let \(E\) be a Banach lattice of two-sided sequences, \((\min(1, 2^k))_{k=-\infty}^{\infty} \in E.\) If \((X_0, X_1)\) is a Banach couple, then the space of the real \(\mathcal{H}\)-method of interpolation \((X_0, X_1)_E\) consists of all \(x \in X_0 + X_1\) such that
\[ \|x\| = \|(\mathcal{H}(2^k, x; X_0, X_1))\|_E < \infty. \tag{1.14} \]
It is readily checked that the space \((X_0, X_1)_E\) is an interpolation space between \(X_0\) and \(X_1\) (cf. [15, page 422]). In the special case \(E = l_p(2^{-k\theta}) (0 < \theta < 1, 1 \leq p \leq \infty)\) we obtain the spaces \((X_0, X_1)_{\theta, p}\) (for the detailed exposition of their properties see [4]).
A couple $\vec{X} = (X_0, X_1)$ is said to be a $\mathfrak{H}$-monotone couple if for every $x \in X_0 + X_1$ and $y \in X_0 + X_1$ there exists a linear operator $U : \vec{X} \to \vec{X}$ such that $y = Ux$ whenever

$$\mathfrak{H}(t, y; X_0, X_1) \leq \mathfrak{H}(t, x; X_0, X_1) \quad \forall t > 0.$$  

(1.15)

As it is well known (cf. [15, page 482]), any interpolation space $X$ with respect to a $\mathfrak{H}$-monotone couple $(X_0, X_1)$ is described by the real $\mathfrak{H}$-method. It means that for some $E$

$$X = (X_0, X_1)_E^\mathfrak{H},$$  

(1.16)

In particular, by the Sparr theorem [20] the couple $(l_1, l_2)$ is a $\mathfrak{H}$-monotone couple. Therefore, if $F$ is an interpolation space between $l_1$ and $l_2$, then there exists $E$ such that

$$F = (l_1, l_2)_E^\mathfrak{H}.$$  

(1.17)

Hence Theorem 1.2 allows to find an r.i.s. that contains Rademacher series with coefficients belonging to an arbitrary interpolation space between $l_1$ and $l_2$. In [19], the similar result was obtained for sequence spaces satisfying more restrictive conditions (see Remark 3.3).

**Theorem 1.4.** Let $F$ be an interpolation sequence space between $l_1$ and $l_2$ and $F = (l_1, l_2)_E^\mathfrak{H}$. Then for the r.i.s. $X = (L_\infty, G)_E^\mathfrak{H}$ we have

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \|a\|_F$$  

(1.18)

with constants independent of $a = (a_k)_{k=1}^\infty$.

Combining Theorem 1.4 with the above remarks, we get the following assertion. If $F$ is a sequence space, then

$$\|a_k\|_F \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X$$  

(1.19)

if and only if $F$ is an interpolation space between $l_1$ and $l_2$.

The last result shows that the restriction of the correspondence (1.13) to interpolation r.i.s. between $L_\infty$ and $G$ is bijective.

**Theorem 1.5.** Let r.i.s.’s $X_0$ and $X_1$ be two interpolation spaces between $L_\infty$ and $G$. If

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_0} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_1},$$  

(1.20)

then $X_0 = X_1$ and the norms of $X_0$ and $X_1$ are equivalent.

In [16, 19], the similar results were obtained by additional conditions with respect to spaces $X_0$ and $X_1$. 

2. Proofs

**Proof of Theorem 1.2.** It is known [10, page 164] that the \( \mathcal{H} \)-functional of a couple of Marcinkiewicz spaces is given by the formula

\[
\mathcal{H}(t,x;M(q_0),M(q_1)) = \sup_{0 < u \leq 1} \frac{\int_0^u x^*(s) \, ds}{\max(q_0(u),q_1(u)/t)}.
\]  

(2.1)

If \( N(t) = \exp(t^2) - 1 \), then the Orlicz space \( L_N \) coincides with the Marcinkiewicz space \( M(q_1) \), where \( q_1(u) = u \log_2^{1/2}(2/u) \) [12]. In addition, \( L_{\infty} = M(q_0) \), where \( q_0(u) = u \). Therefore,

\[
\mathcal{H}(t,x;L_{\infty},G) = \sup_{0 < u \leq 1} \left\{ \frac{1}{u} \int_0^u x^*(s) \, ds \min \left( 1, t \log_2^{-1/2} \left( \frac{2}{u} \right) \right) \right\} \quad \text{for } x \in G.
\]  

(2.2)

Since \( x^*(u) \leq 1/u \int_0^u x^*(s) \, ds \), then from (2.2) it follows that

\[
\mathcal{H}(t,x;L_{\infty},G) \geq \sup_{k=0,1,\ldots} \{ x^*(2^{-k}) \min (1, t(k+1)^{-1/2}) \}.
\]  

(2.3)

Hence,

\[
\mathcal{H}(t,x;L_{\infty},G) \geq x^*(2^{-k_t}) \quad \text{for } t \geq 1,
\]  

(2.4)

where \( k_t = [t^2] - 1 \) ([z] is the integral part of a number z).

Now let \( a = (a_k)_{k=1}^\infty \in l_2 \) and \( x(t) = TA(t) = \sum_{k=1}^\infty A_k(t) \). By the Holmstedt formula [7],

\[
q_0(t) \leq \sum_{k=1}^{[t^2]} a_k^2 + t \left\{ \sum_{k=[t^2]+1}^\infty (a_k^2) \right\}^{1/2} \leq B q_0(t),
\]  

(2.5)

where \( q_0(t) = \mathcal{H}(t,a;l_1,l_2) \), \( (a_k^2)_{k=1}^\infty \) is a nonincreasing rearrangement of the sequence \( (|a_k|)_{k=1}^\infty \), and \( B > 0 \) is a constant independent of \( a = (a_k)_{k=1}^\infty \) and \( t > 0 \).

Assume, at first, that \( a \notin l_1 \). Then inequality (2.5) shows that

\[
\lim_{t \to 0^+} q_0(t) = 0, \quad \lim_{t \to \infty} q_0(t) = \infty.
\]  

(2.6)

The function \( q_0(t) \) belongs to the class \( \mathcal{P} \) [4, page 55]. Therefore it maps the semiaxis \( (0,\infty) \) onto \( (0,\infty) \) one-to-one, and there exists the inverse function \( q_0^{-1} \). By Theorem 1.3, we have

\[
x^*(s) \geq q_0^{-1}(s) \quad \text{for } 0 < s < A^{-1}.
\]  

(2.8)

Obviously, by condition \( t \geq C_1 = C_1(A) = \sqrt{2 \log_2(2A)} \), it holds

\[
2^{-k_t/2} < A^{-1} \quad \text{for } k_t = [t^2] - 1.
\]  

(2.9)

Hence (2.4) and (2.8) imply

\[
\mathcal{H}(t,x;L_{\infty},G) \geq \psi^{-1}(2^{-k_t}).
\]  

(2.10)
Combining the definition of the function $\psi$ with (2.9), we obtain
\[
\psi^{-1}(2^{-k_l}) = A^{-1} \varphi_a(A^{-1/2} \ln^{1/2} (A^{-1}2^{k_l})) \geq A^{-1} \varphi_a \left( \sqrt{\frac{k_l \ln^2 2}{2A}} \right),
\]
(2.11)

From the inequality $t \geq C_1 \geq \sqrt{2}$ it follows that
\[
\frac{\sqrt{k_l}}{t} \geq \sqrt{\frac{\left[ t^2 \right]^{-1}}{\left[ t^2 \right] + 1}} \geq 3^{-1/2}.
\]
(2.12)

Therefore, by (2.10), we have
\[
\mathcal{K}(t, x; L_\infty, G) \geq C_2 \varphi_a(t) \quad \text{for } t \geq C_1,
\]
(2.13)

where $C_2 = C_2(A) = \sqrt{\ln 2/6}A^{-3/2}$.

If now $t \geq 1$, then the concavity of the $\mathcal{K}$-functional and the previous inequality yield
\[
\mathcal{K}(t, x; L_\infty, G) \geq C_1^{-1} \mathcal{K}(tC_1, x; L_\infty, G) \geq C_2 \varphi_a(C_1 t) \geq C_2 \varphi_a(t).
\]
(2.14)

Using the inequalities $\|a\|_2 \leq \|a\|_1$ ($a \in l_1$) and $\|x\|_G \leq \|x\|_\infty$ ($x \in L_\infty$), the definition of the $\mathcal{K}$-functional, and Theorem 1.1, we obtain
\[
\mathcal{K}(t, x; L_\infty, G) = t \|x\|_G \geq C_3 t \|a\|_2 = C_3 \varphi_a(t) \quad \text{for } 0 < t \leq 1.
\]
(2.15)

Thus,
\[
\mathcal{K}(t, a; l_1, l_2) \leq C \mathcal{K}(t, Ta; L_\infty, G),
\]
(2.16)

if $C = \max(C_2^{-1}, C_1/C_2)$.

Suppose now $a \in l_1$. By (2.5), without loss of generality, we can assume that the function $\varphi_a$ maps the semiaxis $[0, \infty)$ injectively onto the interval $(0, \|a\|_1)$. Hence we can define the mappings $\varphi^{-1}_a : (0, \|a\|_1) \to (0, \infty)$, $\psi : (0, A^{-1} \|a\|_1) \to (0, A^{-1})$, and $\psi^{-1} : (0, A^{-1}) \to (0, A^{-1} \|a\|_1)$. Arguing as above, we get inequality (2.16).

The opposite inequality follows from Theorem 1.1 and relation (1.4). Indeed,
\[
\mathcal{K}(t, Ta; L_\infty, G) \leq \inf \left\{ \|Ta^0\|_\infty + t \|Ta^1\|_G : a = a^0 + a^1, a^0 \in l_1, a^1 \in l_2 \right\} \\
\leq D \mathcal{K}(t, a; l_1, l_2).
\]
(2.17)

**Proof of Theorem 1.4.** It is sufficient to use Theorem 1.2 and the definition of the real $\mathcal{K}$-method of interpolation. 

For the proof of Theorem 1.5 we need some definitions and auxiliary assertions. These results are also of some independent interest.

Let $f(t)$ be a function defined on the interval $(0, l)$, where $l = 1$ or $l = \infty$. Then the dilation function of $f$ is defined as follows:
\[
M_f(t) = \sup \left\{ \frac{f(st)}{f(s)} : s, st \in (0, l) \right\}, \quad \text{if } t \in (0, l).
\]
(2.18)
Since this function is semimultiplicative, then there exist numbers
\[
\gamma_f = \lim_{t \to 0^+} \frac{\ln M_f(t)}{\ln t}, \quad \delta_f = \lim_{t \to \infty} \frac{\ln M_f(t)}{\ln t}.
\]

A Banach couple \( \vec{X} = (X_0, X_1) \) is called a partial retract of a couple \( \vec{Y} = (Y_0, Y_1) \) if each element \( x \in X_0 + X_1 \) is orbitally equivalent to some element \( y \in Y_0 + Y_1 \). The last means that there exist linear operators \( U : \vec{X} \to \vec{Y} \) and \( V : \vec{Y} \to \vec{X} \) such that \( Ux = y \) and \( Vy = x \).

**Proposition 2.1.** Suppose that \( M(\varphi) \) is a Marcinkiewicz space on \([0,1]\). If \( \gamma_\varphi > 0 \), then \( \vec{X} = (L_\infty, M(\varphi)) \) is a \( \mathcal{H}\)-monotone couple.

**Proof.** It is sufficient to show that the couple \( \vec{X} \) is a partial retract of the couple \( \vec{Y} = (L_\infty, L_\infty(\tilde{\varphi})) \), where
\[
\|x\|_{L_\infty(\tilde{\varphi})} = \sup_{0 < t \leq 1} \tilde{\varphi}(t) |x(t)|, \quad \tilde{\varphi}(t) = \frac{t}{\varphi}(t).
\]

Indeed, a partial retract of a \( \mathcal{H}\)-monotone couple is a \( \mathcal{H}\)-monotone couple [15, page 420], and by the Sparr theorem [20] \( \vec{Y} \) is a \( \mathcal{H}\)-monotone couple.

First note that the inclusion \( L_\infty \subset M(\varphi) \) implies \( L_\infty + M(\varphi) = M(\varphi) \). So, let \( x \in M(\varphi) \). Without loss of generality [10, page 87], assume that \( x(t) = x^*(t) \). Define the operator
\[
U_1 y(t) = \sum_{k=1}^{\infty} 2^{k-1} \int_0^{2^{-k}} y(s) dx_{(2^{-k}, 2^{-k}+1)}(t) \quad \text{for} \ y \in M(\varphi).
\]

Clearly, \( U_1 \) maps \( L_\infty \) into itself. In addition, the concavity of the function \( \varphi \) and properties of the nonincreasing rearrangement imply
\[
\|U_1 y\|_{L_\infty(\tilde{\varphi})} \leq 2 \sup_{k=1,2,...} (\varphi(2^{-k+1}))^{-1} \int_0^{2^{-k}} y^*(s) ds \leq 2 \|y\|_{M(\varphi)}.
\]

Hence \( U_1 : \vec{X} \to \vec{Y} \). Since \( x(t) \) is nonincreasing, then \( U_1 x(t) \geq x(t) \). Therefore the linear operator
\[
Uy(t) = \frac{x(t)}{U_1 x(t)} U_1 y(t)
\]
is bounded from the couple \( \vec{X} \) into the couple \( \vec{Y} \). In addition, \( Ux(t) = x(t) \).

Take for \( V \) the identity mapping, that is, \( Vy(t) = y(t) \). Since \( \gamma_f > 0 \), then, by [10, page 156], we have
\[
\|Vy\|_{M(\varphi)} \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) y^*(t) \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) |y^*(t)| = C \|y\|_{L_\infty(\tilde{\varphi})}.
\]

Therefore \( V : \vec{Y} \to \vec{X} \) and \( VX = x \).

Thus an arbitrary element \( x \in M(\varphi) \) is orbitally equivalent to itself as to element of the space \( L_\infty + L_\infty(\tilde{\varphi}) \). This completes the proof.
**Corollary 2.2.** If \( \gamma_0 > 0 \), then \((L_\infty, M(\varphi)^0)\) is a \( \mathcal{H} \)-monotone couple.

**Proof.** Assume that \( x \) and \( y \) belong to the space \( M(\varphi)^0 \) and
\[
\mathcal{H}(t, y; L_\infty, M(\varphi)^0) \leq \mathcal{H}(t, x; L_\infty, M(\varphi)^0) \quad \text{for } t > 0.
\]
(2.25)

If \( z \in M(\varphi)^0 \), then
\[
\mathcal{H}(t, z; L_\infty, M(\varphi)^0) = \mathcal{H}(t, z; L_\infty, M(\varphi)).
\]
(2.26)

Therefore, \( \mathcal{H}(t, y; L_\infty, M(\varphi)) \leq \mathcal{H}(t, x; L_\infty, M(\varphi)) \) for \( t > 0 \).
(2.27)

Hence, by Proposition 2.1, there exists an operator \( T : (L_\infty, M(\varphi)) \to (L_\infty, M(\varphi)) \) such that \( y = Tx \). It is readily seen that \( M(\varphi)^0 \) is an interpolation space of the couple \((L_\infty, M(\varphi))\). Therefore \( T : (L_\infty, M(\varphi)^0) \to (L_\infty, M(\varphi)^0) \).

We define now two subcones of the cone \( \mathcal{P} \). Denote by \( \mathcal{P}_0 \) the set of all functions \( f \in \mathcal{P} \) such that \( \lim_{t \to 0} f(t) = \lim_{t \to \infty} f(t)/t = 0 \). If \( f \in \mathcal{P}_0 \), then \( 0 \leq \gamma_f \leq \delta_f \leq 1 \) [10, page 76]. Let \( \mathcal{P}^{++} \) be the set of all \( f \in \mathcal{P} \) such that \( 0 < \gamma_f \leq \delta_f < 1 \). It is obvious that \( \mathcal{P}^{++} \subset \mathcal{P}_0 \).

A couple \((X_0, X_1)\) is called a \( \mathcal{H}_0 \)-complete couple if for any function \( f \in \mathcal{P}_0 \) there exists an element \( x \in X_0 + X_1 \) such that
\[
\mathcal{H}(t, x; X_0 + X_1) = f(t).
\]
(2.28)

In other words, the set \( \mathcal{H}(X_0 + X_1) \) of all \( \mathcal{H} \)-functionals of a \( \mathcal{H}_0 \)-complete couple \((X_0, X_1)\) contains, up to equivalence, the whole of the subcone \( \mathcal{P}_0 \).

**Proposition 2.3.** The Banach couple \((L_1(0, \infty), L_2(0, \infty))\) is a \( \mathcal{H}_0 \)-complete couple.

**Proof.** By the Holmstedt formula for functional spaces [7],
\[
\mathcal{H}(t, x; L_1, L_2) \asymp \max \left\{ \int_0^t x^*(s) ds, t \left[ \int_t^\infty (x^*(s))^2 ds \right]^{1/2} \right\}.
\]
(2.29)

If \( f \in \mathcal{P}_0 \), then \( g(t) = f(t^{1/2}) \) belongs to \( \mathcal{P}_0 \). We denote \( x(t) = g'(t) \). Then \( x(t) = x^*(t) \) and
\[
\int_0^t x(s) ds = g(t).
\]
(2.30)

Assume that \( f \in \mathcal{P}^{++} \). If \( \delta_f < 1 \), then there exists \( \varepsilon > 0 \) such that for some \( C > 0 \)
\[
G(s) = f(s^{1/2}) \leq C \left( \frac{3}{2} \right)^{1-\varepsilon} f(t^{1/2}), \quad \text{if } s \geq t.
\]
(2.31)

Since \( g \in \mathcal{P}_0 \), then \( g'(t) \leq g(t)/t \). Therefore for \( t > 0 \)
\[
\int_t^\infty (x(s))^2 ds \leq \int_t^\infty \frac{g^2(s)}{s^2} ds \leq C^2 t^{\varepsilon-1} (f(t^{1/2}))^2 \int_t^\infty s^{-1-\varepsilon} ds = C^2 \varepsilon t^{-1} (g(t))^2.
\]
(2.32)

Combining this with (2.29) and (2.30), we obtain
\[
\mathcal{H}(t, x; L_1, L_2) \asymp g(t^2) = f(t).
\]
(2.33)
Thus $\mathcal{K}(L_1 + L_2) \supset \mathcal{P}^+$, Hence, in particular, the intersection $\mathcal{K}(X_0 + X_1) \cap \mathcal{P}^+$ is not empty. Therefore, by \cite[Theorem 4.5.7]{6}, $(L_1, L_2)$ is a $\mathcal{K}_0$-complete Banach couple. This completes the proof. 

Let $\mathcal{K}(l_1 + l_2)$ be the set of all $\mathcal{K}$-functionals corresponding to the couple $(l_1, l_2)$. By $\mathcal{F}$ we denote the set of all functions $f \in \mathcal{P}$ such that

$$f(t) = f(1)t \quad \text{for} \ 0 < t \leq 1, \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0.$$  

(2.34)

**COROLLARY 2.4.** Up to equivalence,

$$\mathcal{K}(l_1 + l_2) \supset \mathcal{F}.$$  

(2.35)

**Proof.** It is well known (cf. \cite[page 142]{4}) that for $x \in L_1(0, \infty) + L_\infty(0, \infty)$ and $u > 0$

$$\mathcal{K}(u, x; L_1, L_\infty) = \int_0^u x^*(s) \, ds.$$  

(2.36)

In addition,

$$L_1 = (L_1, L_\infty)^{\mathcal{K}}_{l_\infty}, \quad L_2 = (L_1, L_\infty)^{\mathcal{K}}_{l_2(2^{-k/2})}.$$  

(2.37)

The spaces $l_\infty$ and $l_2(2^{-k/2})$ are interpolation spaces with respect to the couple $(l_\infty, l_2(2^{-k}))$ \cite{4}. Therefore, by the reiteration theorem (see \cite{5} or \cite{14}),

$$\mathcal{K}(t; x; L_1, L_2) \approx \mathcal{K}(t; x; L_1, L_\infty) \approx \mathcal{K}(t; x; L_1, L_\infty)$$  

(2.38)

for $x \in L_1 + L_2$. 

Introduce the average operator:

$$Qx(t) = \sum_{k=1}^\infty \int_{k-1}^k x(s) \, ds \chi(k, k], \quad \text{if} \ t > 0.$$  

(2.39)

From (2.36) it follows that

$$\mathcal{K}(t; Qx^*; L_1, L_\infty) = \mathcal{K}(t; x; L_1, L_\infty)$$  

(2.40)

for all positive integers $t$. Both functions in (2.40) are concave. Therefore,

$$\mathcal{K}(t; Qx^*; L_1, L_\infty) \approx \mathcal{K}(t; x; L_1 \cdot L_\infty) \quad \forall t \geq 1.$$  

(2.41)

Hence (2.38) yields

$$\mathcal{K}(t; Qx^*; L_1, L_\infty) \approx \mathcal{K}(t; x; L_1, L_2), \quad \text{if} \ t \geq 1.$$  

(2.42)

Now let $f \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{P}_0$, then, by Proposition 2.3, there exists a function $x \in L_1(0, \infty) + L_2(0, \infty)$ such that

$$\mathcal{K}(t; x; L_1, L_2) \approx f(t).$$  

(2.43)

Clearly, the operator $Q$ is a projector in the spaces $L_1$ and $L_2$ with norm 1. Moreover, $Q(L_1) = l_1$ and $Q(L_2) = l_2$. Hence, by the theorem about complemented subcouples
mentioned in Section 1 (see [3] or [21, page 136]),
\[ \mathcal{K}(t, Qx^*; L_1, L_2) \approx \mathcal{K}(t, a; l_1, l_2) \quad \text{for } t > 0, \] (2.44)
where \( a = (\int_{k=1}^{\infty} x^*(s) \, ds)_{k=1}^{\infty} \).

Thus (2.42) and (2.43) imply
\[ \mathcal{K}(t, a; l_1, l_2) \approx f(t) \quad \text{for } t \geq 1. \] (2.45)
The last relation also holds if \( 0 < t \leq 1 \). Indeed, in this case
\[ \mathcal{K}(t, a; l_1, l_2) \approx t \parallel a \parallel_2 = t \parallel a \parallel_{l_2} \approx t f(1) = f(t). \] (2.46)
This completes the proof.

**Proof of Theorem 1.5.** As it was already mentioned in the proof of Theorem 1.2, the Orlicz space \( L_N, N(t) = \exp(t^2) - 1 \), coincides with the Marcinkiewicz space \( M(\varphi_1) \), for \( \varphi_1(u) = u \log^{1/2}(2/u) \). Since \( \gamma_{\varphi_1} = 1 \), then Corollary 2.2 implies that the couple \( (L_\infty, G) \) is a \( \mathcal{K} \)-monotone couple. Hence,
\[ X_0 = (L_\infty, G)^{\mathcal{K}}_{E_0}, \quad X_1 = (L_\infty, G)^{\mathcal{K}}_{E_1}, \] (2.47)
for some parameters of the real \( \mathcal{K} \)-method of interpolation \( E_0 \) and \( E_1 \). By Theorem 1.4,
\[ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_i} \approx \left\| (a_k) \right\|_{F_i}, \] (2.48)
where \( F_i = (l_1, l_2)^{\mathcal{K}}_{E_i} \) (i = 0, 1). So
\[ (l_1, l_2)^{\mathcal{K}}_{E_0} = (l_1, l_2)^{\mathcal{K}}_{E_1}. \] (2.49)
Equation (2.49) means that the norms of spaces \( E_0 \) and \( E_1 \) are equivalent on the set \( \mathcal{K}(l_1 + l_2) \). It is readily to check that this set coincides, up to the equivalence, with the set \( \mathcal{K}(L_\infty + G) \) of all \( \mathcal{K} \)-functionals corresponding to the couple \( (L_\infty, G) \). More precisely,
\[ \mathcal{K}(l_1 + l_2) = \mathcal{K}(L_\infty + G) = \mathcal{F}. \] (2.50)
In fact, by Theorem 1.2 and Corollary 2.2, \( \mathcal{F} \subset \mathcal{K}(l_1 + l_2) \subset \mathcal{K}(L_\infty + G) \). On the other hand, since \( L_\infty \subset G \) with the constant 1 and \( L_\infty \) is dense in \( G \), then \( \mathcal{K}(L_\infty + G) \subset \mathcal{F} \) [15, page 386].

Now let \( x \in X_0 \). By (2.47), we have \( (\mathcal{K}(2^k, x; L_\infty, G))_{k} \in X_0 \). Using (2.50), we can find \( a \in l_2 \) such that
\[ \mathcal{K}(2^k, a; l_1, l_2) \approx \mathcal{K}(2^k, x; L_\infty, G) \] (2.51)
for all positive integers \( k \). Since a parameter of \( \mathcal{K} \)-method is a Banach lattice, then this implies \( (\mathcal{K}(2^k, a; l_1, l_2))_{k} \in E_0 \). Therefore, by (2.49), \( (\mathcal{K}(2^k, a; l_1, l_2))_{k} \in E_1 \), that is, \( (\mathcal{K}(2^k, x; L_\infty, G))_{k} \in E_1 \) for \( x \in X_1 \). Thus \( X_0 \subset X_1 \). Arguing as above, we obtain the converse inclusion, and \( X_0 = X_1 \) as sets. Since \( X_0 \) and \( X_1 \) are Banach lattices, then their norms are equivalent. This completes the proof. \( \square \)
3. Final remarks and examples

**Remark 3.1.** Combining Theorems 1.2, 1.4, and 1.5 with results obtained in [8], we can prove similar assertions for lacunary trigonometric series. Moreover, taking into account the main result of [1], we can extend Theorems 1.2, 1.4, and 1.5 to Sidon systems of characters of a compact abelian group.

**Remark 3.2.** In Theorem 1.2, we cannot replace the space $G$ by $L_q$ with some $q < \infty$. Indeed, suppose that the couple $(T(l_1), T(l_2))$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, L_q)$, that is,

$$\mathcal{K}(t, a; l_1, l_2) \approx \mathcal{K}(t, Ta; L_\infty, L_q).$$

Let $E = l_p(2^{-\theta k})$, where $0 < \theta < 1$ and $p = q/\theta$. Applying the $\mathcal{K}$-method of interpolation $(\cdot, \cdot)_E$ to the couples $(l_1, l_2)$ and $(L_\infty, L_q)$, we obtain

$$\|Ta\|_p \approx \|a\|_{r,p} = \left\{ \sum_{k=1}^{\infty} (a_k^*)^p k^{p/r-1} \right\}^{1/p}.$$  

(3.2)

Since $r = 2/(2-\theta) < 2$ [4, page 142], then this contradicts with (1.3).

**Remark 3.3.** Clearly, a partial retract of a couple $\vec{Y} = (Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $\vec{Y}$. The opposite assertion is not true, in general (nevertheless, some interesting examples of $\mathcal{K}$-subcouples and partial retracts simultaneously are given in [9]). Indeed, by Theorem 1.2, the subcouple $(l_1, l_2)$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, G)$. Assume that $(l_1, l_2)$ is a partial retract of this couple. Then (see the proof of Proposition 2.1) $(l_1, l_2)$ is a partial retract of the couple $(L_\infty, L_\infty(\log_2^{1/2}(2/t)))$, as well. Therefore, by Lemma 1 from [2] and [4, page 142] it follows that

$$[l_1, l_2]_\theta = (l_1, l_2)_{\theta, \infty} = l_p, \infty,$$

(3.3)

where $[l_1, l_2]_\theta$ is the space of the complex method of interpolation [4], $0 < \theta < 1$, and $p = 2/(2-\theta)$. On the other hand, it is well known [4, page 139] that

$$[l_1, l_2]_\theta = l_p \quad \text{for} \quad p = \frac{2}{2-\theta}.$$  

(3.4)

This contradiction shows that the couple $(l_1, l_2)$ is not a partial retract of the couple $(L_\infty, G)$.

Using Theorem 1.4, we can find coordinate sequence spaces of coefficients of Rademacher series belonging to certain r.i.s.’s.

**Example 3.4.** Let $X$ be the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t \log_2 \log_2 (16/t)$, $0 < t \leq 1$. Show that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)} \approx \|a\|_{l_1(\log)},$$

(3.5)

where $l_1(\log)$ is the space of all sequences $a = (a_k)_{k=1}^{\infty}$ such that the norm

$$\|a\|_{l_1(\log)} = \sup_{k=1,2,\ldots} \log_2^{-1}(2k) \sum_{i=1}^{k} a_i.$$  

(3.6)
is finite. Taking into account Theorem 1.4, it is sufficient to check that

\begin{equation}
(l_1, l_2)_F^{\mathfrak{K}} = l_1(\log),
\end{equation}

\begin{equation}
(l_\infty, G)_F^{\mathfrak{K}} = M(\varphi),
\end{equation}

for some parameter \( F \) of the \( \mathfrak{K} \)-method of interpolation. More precisely, we will prove that (3.7) and (3.8) are true for \( F = l_\infty(u_k) \), where \( u_k = 1/(k + 1) \) \((k \geq 0)\) and \( u_k = 1 \) \((k < 0)\).

By the Holmstedt formula (2.5),

\begin{equation}
\varphi^a_{2k} \leq \sum_{i=1}^{2k} \varphi^*_i + 2^k \left[ \sum_{i=2^{2k+1}}^{\infty} \left( \varphi^*_i \right)^2 \right]^{1/2} \leq B\varphi^a_{2k} \quad \text{for } k = 0, 1, 2, \ldots,
\end{equation}

where, as before, \( \varphi^a(t) = \mathfrak{K}(t, a; l_1, l_2) \). Without loss of generality, assume that \( a_i = a^*_i \). If \( \|a\|_{l_1(\log)} = R < \infty \), then by (3.6),

\begin{equation}
\sum_{i=1}^{2k} \varphi^*_i \leq 2R(k + 1).
\end{equation}

In particular, this implies \( \varphi^a_{2k} \leq 2^{-2k+1}R(k + 1) \), for nonnegative integer \( k \). Using (3.10), we obtain

\begin{equation}
\sum_{i=2^{2k+1}}^{\infty} \varphi^*_i \leq \sum_{j=0}^{2^k} \sum_{i=2^{2j+1}}^{2^{2j+1}} \varphi^*_i \leq 3 \sum_{j=0}^{\infty} 2^j \varphi^*_i \leq 12R \sum_{j=0}^{\infty} 2^{-2j} \leq 12R \sum_{j=0}^{\infty} 2^{-2j} \leq 144R(k + 1)^2 2^{-2k}.
\end{equation}

Hence the second term in (3.9) does not exceed \( 12R(k + 1) \). Therefore, if \( E = (l_1, l_2)_F^{\mathfrak{K}} \), then (3.10) implies

\begin{equation}
\|a\|_E = \sup_{k=0, 1, \ldots} \varphi^a_{2k} \leq 14\|a\|_{l_1(\log)}.
\end{equation}

Conversely, if \( 2^j \leq k \leq 2^{j+1} \) for some \( j = 0, 1, 2, \ldots \), then from (3.9) it follows that

\begin{equation}
\sum_{i=1}^{k} \varphi^*_i \leq B\varphi^a_{2(j+1)} \leq \sum_{i=1}^{2^{2(j+1)}} \varphi^*_i \leq 2^{j+2} \log_2(2k) \|a\|_E.
\end{equation}

Therefore, \( \|a\|_{l_1(\log)} \leq 2B\|a\|_E \) and (3.7) is proved.

We pass now to function spaces. At first, we introduce one more interpolation method which is, actually, a special case of the real method of interpolation. For a function \( \varphi \in \mathfrak{P} \) and an arbitrary Banach couple \((X_0, X_1)\) define generalized Marcinkiewicz space as follows:

\begin{equation}
M_\varphi(X_0, X_1) = \left\{ x \in X_0 + X_1 : \sup_{t > 0} \frac{\mathfrak{K}(t, x; X_0, X_1)}{\varphi(t)} < \infty \right\}.
\end{equation}
Let \( q_0(t) = \min(1,t), \ q_1(t) = \min(1,t\log_2^{1/2}[\max(2,2/t)]), \) and \( N(t) = \exp(t^2) - 1, \) as before. By equation (2.36), we have

\[
\begin{align*}
L_\infty &= M_{q_0}(L_1,L_\infty), \\
L_\infty &= M_{q_1}(L_1,L_\infty),
\end{align*}
\]

(here \( L_\infty \) and \( L_N \) are functional spaces on the segment \([0,1]\)). In addition, using similar notation, it is easy to check that

\[
(X_0,X_1)_F^\infty = M_\rho (X_0,X_1),
\]

for an arbitrary Banach couple \((X_0,X_1)\) and \( \rho(t) = \log_2(4 + t) \). Hence, by the reiteration theorem for generalized Marcinkiewicz spaces [15, page 428], we obtain

\[
(L_\infty,L_N)_F^\infty = M_\rho (M_{q_0}(L_1,L_\infty),M_{q_1}(L_1,L_\infty)) = M_{q_0}(L_1,L_\infty) = M(\rho),
\]

where \( \rho(t) = q_0(t)\rho(q_1(t)/q_0(t)) \). A simple calculation gives \( q_0(t) \times q(t) \), if \( t > 0 \). Thus,

\[
(L_\infty,L_N)_F^\infty = M(\rho).
\]

It is readily seen that \( \mathcal{H}(t,x;L_\infty,G) = \mathcal{H}(t,x;L_\infty,L_N), \) for all \( x \in G \). Therefore, for such \( x \) the norm \( \|x\|_{M(\rho)} \) is equal to the norm \( \|x\|_Y \), where \( Y = (L_\infty,G)_F^\infty \). On the other hand, for \( x \in M(\rho) \)

\[
\frac{1}{t\log_2^{1/2}(2/t)} \int_0^t x^*(s) ds \leq \|x\|_{M(\rho)} \frac{\log_2\log_2(16/t)}{\log_2^{1/2}(2/t)} \to 0 \quad \text{as} \quad t \to 0 +.
\]

This implies that \( M(\rho) \subset G \) [10, page 156]. Thus \( Y = M(\rho) \), and (3.8) is proved. Equivalence (3.5) follows now, as already stated, from (3.7) and (3.8).

**Remark 3.5.** Theorems 1.4 and 1.5 strengthen results of [18, 19], where similar assertions are obtained for sequence spaces \( F \) satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator

\[
\sigma_n a = \left( a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots \right)
\]

in the space \( l_1(\ln) \) (see Example 3.6) is equal to \( n \). Therefore, condition (11) from [19] fails for this space and the theorems obtained in [18, 19] cannot be applied to it. Similarly, the Marcinkiewicz space \( M(\rho) \) from Example 3.4 does not satisfy the conditions of Theorem 8 of [19].

Using Theorems 1.4 and 1.5, we can derive certain interpolation relations.

**Example 3.6.** Let \( \varphi \in \mathcal{P} \) and \( 1 \leq p < \infty \). Recall that the Lorentz space \( \Lambda_p(\varphi) \) consists of all measurable functions \( x = x(s) \) such that

\[
\|x\|_{\varphi,p} = \left\{ \int_0^1 (x^*(s))^p d\varphi(s) \right\}^{1/p} < \infty.
\]
In [19], V. A. Rodin and E. M. Semenov proved that
\[ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\varphi,p} \leq \| (a_k) \|_{\varphi,p}, \] (3.22)
where \( \varphi(s) = \log_2^{1-p} (2/s) \) and \( 1 < p < 2 \). Moreover, the space \( \Lambda_p(\varphi) \) is the unique r.i.s. having this property. Note that \( l_p = (l_1, l_2)_{\theta,p} \), where \( \theta = 2(p-1)/p \) [4, page 142].
Therefore, by Theorem 1.4, we obtain
\[ (L_\infty, G)_{\theta,p} = \Lambda_p(\varphi) \] (3.23)
for the same \( p \) and \( \theta \).

ACKNOWLEDGEMENT. The author is grateful to Prof. S. Montgomery-Smith for useful advices and to referees for their suggestions and remarks.

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Sergey V. Astashkin: Department of Mathematics, Samara Street University, Academic Pavlov Street, 1, Samara, 443011, Russia

E-mail address: astashkn@ssu.samara.ru