A NEW COMBINATORIAL IDENTITY

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ABSTRACT. We prove a combinatorial identity which arose from considering the relation
\[ r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p) \]
in connection with Fermat's last theorem.

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The following combinatorial identity:
\[
\sum_{l'=0}^{l} \sum_{j'=0}^{j} \frac{1}{(m-l') \binom{m-l'+j'-1}{j'}} \frac{m-l'}{j'} \binom{m-l'}{2(l-l')-(j-j')} \binom{m-l'}{j-j'} \binom{m-l}{2l+1} \binom{2l+1}{j} = \frac{1}{2(m-l)} \binom{2m}{2l+1} \binom{2l+1}{j} = \frac{1}{2(m-l)} \binom{2m}{2l} \binom{2l+1}{j} (1)
\]

for all \( m > l \geq 0 \), where \( m, l, \) and \( j \) are nonnegative integers and \( 0 \leq j \leq 2l + 1 \), arose from considering
\[ r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p) \]

in connection with Fermat's last theorem (FLT), which was proved in 1994 by Wiles and Taylor. Recall that FLT states that \( x^p + y^p - z^p \neq 0 \), where \( x, y, z, p \) are any nonzero integers and \( p > 2 \). We take, without loss of generality, that \( x, y, \) and \( z \) are relatively prime and \( p \) is prime. In general, \( r_p(x, y, z) \) can be factored as \( p(z-x)(z-y)(x+y)f_p(x, y, z) \) which are powers of \( p \) if \( x^p + y^p - z^p = 0 \). These factors result in the elementary Abel-Barlow relations known since the 1820's (see [2]).

However, the last factor \( f_p(x, y, z) \) is
\[
\sum_{l=0}^{m-1} \sum_{i=0}^{2l} \sum_{j=0}^{i} \frac{(-1)^{l-j} \binom{m+l-j}{2l-j+1} \binom{m-l+j-1}{j}}{(m-l)} x^{2l-i} y^i (z-x)^{m-l-1} (z-y)^{m-l-1} = \begin{cases} p^{kp-1} d^p, & \text{if } p \mid xyz, \\ d^p, & \text{otherwise} \end{cases} (3)
\]

where \( p = 2m + 1 \geq 5 \) and \( k > 0 \). This formulation of \( f_p(x, y, z) \), which is believed to be novel, establishes the new identity. However, it appears to offer no new insights into a possible elementary proof of FLT.
To discover the identity, note that
\[ r_p(x, y, z) = p \sum_{l=0}^{m} \sum_{j=0}^{m} \frac{(-1)^l}{p} \binom{p-l}{j} x^j y^{p-j-l} z^l, \] (4)
where \( j + l \neq 0 \).

Alternatively, we have
\[ r_p(x, y, z) = p \sum_{l'=0}^{m} (z-x)^{m-l'} (z-y)^{m-l'} \sum_{j'=0}^{2l'+1} a_{j', m-l'} x^{2l'-j'+1} y^{j'}. \] (5)

Equating (4) and (5) for a given \( j \) and \( l \), we get the recurrence
\[ a_{j, m-l} = \frac{1}{2(m-l)} \binom{2m}{2l+1} \binom{2l+1}{j} - \sum_{l'<j'} \sum_{j'=j} \binom{m-l'}{j-j'} \binom{m-l'}{j'} a_{j', m-l'}. \] (6)

Now,
\[ a_{j, m-l} = \frac{1}{2(m-l)} \binom{m-l-j}{2l-j+1} \binom{m-l+j-1}{j} \] (7)
satisfies the recurrence (6). Substituting the expression for \( a_{j, m-l} \) and rearranging, we obtain the new identity.

The authors have reviewed the literature, notably Gould [1] and Riordan [3] as well as the relevant journals since 1980. Based on this review, (1) is believed to be novel.

**Proof of the identity.** We consider two special cases.

**Case 1** \((j = 0)\). Equation (1) reduces to:
\[ \sum_{l' \leq l} \frac{1}{m-l'} \binom{m+l'}{2l'+1} \binom{m-l'}{2l-2l'} = \frac{1}{2l+1} \binom{2m}{2l}. \] (8)

Divide both sides of (8) by the right-hand side and denote the resulting left-hand side by \( S(m, l) \). Then \( S(m, l) \) satisfies the recurrence equation \( S(m+1, l) - S(m, l) = 0 \)—obtained by using Zeilberger’s [5] Ekhad, a computer algebra package which is available from http://www.math.temple.edu/~zeilberg/—and hence the identity follows from the fact that \( S(1, 1) = 1 \).

**Case 2** \((j \neq 0)\). Equation (1) reduces to
\[ \sum_{l'} \sum_{j'} \binom{m-l'+j'}{2l'-j'+1} \binom{m-l'}{j-1} \binom{m-l'}{j-j'} \binom{2l-1}{j-j'} \binom{2l}{j} = \binom{2m}{2l} \binom{2l}{j-1}, \] (9)

which by multiplying both sides by \((2l-j+1)/j\) is also expressible as
\[ \sum_{l'} \sum_{j'} \binom{m-1-l'+j'}{2l'-j'+1} \binom{m-l'}{j-j'} \binom{m-l'}{2l-j} \binom{2l-j+1}{2l'-j'+1} = \binom{2m}{2l} \binom{2l}{j} = \binom{2m}{j, 2l-j}, \] (10)

where
\[ \binom{a}{b, c} := \frac{a!}{b!c!(a-b-c)!}. \] (11)
Equation (10) follows from the identity
\[
\sum_{l'} \sum_{j'} \left( p - 1 - l' + j' \right) \left( p - l' \right) \left( m + l' - j' \right) \left( m + l' \right) = \binom{m + p}{j, k}
\] (12)
with \( p = m \) and \( k = 2l - j \).

Denote the left-hand side of (12) by \( S(m, p, j, k) \). \( S(m, p, j, k) \) satisfies \( S(m+1, p, j, k) = S(m, p, j, k) \) and hence \( S(m, p, j, k) = S(m+p, 0, j, k) \). Hence to prove (12) it suffices to prove
\[
S(n, 0, j, k) = \binom{n}{j, k} \quad \forall n, j, k \in \mathbb{Z}_\geq 0.
\] (13)

Clearly (13) is true for \( n = 0 \). Now, let \( n > 0 \) and set \( S(n, j, k) := S(n, 0, j, k) \). Then \( S(n, j, k) \) satisfies the recurrence equation
\[
(-1 + j - n)S(n-1, j-1, k) - (1 + k)S(n-1, j, k-1) \\
+ (j - k - n - 1)S(n-1, j, k) + (j + k - n - 1)S(n, j-1, k) \\
+ (k + 1)S(n, j-1, k+1) + (j + 2k - n + 1)S(n, j, k) + 2(1 + k)S(n, j, k+1) = 0
\] (14)
that is obtained by using Wegschaider’s [4] MultiSum, a computer algebra package which is available from http://www.risc.uni-linz.ac.at/research/combinat/risc/software/. Note that the right-hand side of (13) also satisfies (14). Hence by induction it follows that
\[
S(n, j, k) = \binom{n}{j, k} \quad \forall n, j, k \in \mathbb{Z}_\geq 0.
\] (15)

REFERENCES


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