COMMON FIXED POINTS OF SET-VALUED MAPPINGS

M. R. SINGH, L. S. SINGH, and P. P. MURTHY

(Received 27 August 1996 and in revised form 8 July 1998)

Dedicated to late P. V. Lakshmaiah

ABSTRACT. The main purpose of this paper is to obtain a common fixed point for a pair of set-valued mappings of Greguš type condition. Our theorem extend Diviccaro et al. (1987), Guay et al. (1982), and Negoescu (1989).

2000 Mathematics Subject Classification. Primary 54H25, 47H10.

1. Introduction. Greguš [4] proved the following result.

Theorem 1.1. Let \( C \) be a closed convex subset of a Banach space \( X \). If \( T \) is a mapping of \( C \) into itself satisfying the inequality

\[
\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\| \tag{1.1}
\]

for all \( x, y \in C \), where \( 0 < a < 1 \), \( 0 \leq c \), \( 0 \leq b \), and \( a + b + c = 1 \), then \( T \) has a unique fixed point in \( C \).

Mappings satisfying the inequality (1.1) with \( a = 1 \) and \( b = c = 0 \) is called nonexpansive and it was considered by Kirk [6], whereas the mapping with \( a = 0 \), \( b = c = 1/2 \) by Wong [13]. Recently, Fisher et al. [3], Diviccaro et al. [2], Mukherjee et al. [9], and Murthy et al. [10] generalized Theorem 1.1 in many ways. In this context, we prove a common fixed point theorem for set-valued mappings using Greguš type condition. Before presenting our main theorem we need the following definitions and lemma for our main theorem.

Let \( (X, d) \) be a metric space and \( CB(X) \) be the class of nonempty closed bounded subsets of \( X \). For any nonempty subsets \( A, B \) of \( X \) we define

\[
D(A, B) = \inf \{ d(a, b) : a \in A, \ b \in B \},
\]

\[
H(A, B) = \max \{ \sup \{ D(a, b) : a \in A \}, \ \sup \{ D(A, b) : b \in B \} \}. \tag{1.2}
\]

The space \( CB(X) \) is a metric space with respect to the above defined distance function \( H \) (see Kuratowski [7, page 214] and Berge [1, page 126]). Nadler [11] has defined the contraction mapping for set-valued mappings. A set-valued mapping \( F : X \rightarrow CB(X) \) is said to be contraction if there exists a real number \( k \), \( 0 \leq k < 1 \) such that \( H(Fx, Fy) \leq k \cdot d(x, y) \), for all \( x, y \in X \).

Throughout this paper \( C(X) \) stands for a class of nonempty compact subset of \( X \), \( D(A, B) \) is the distance between two sets \( A \) and \( B \).

The following Definitions 1.2, 1.3, 1.4, and 1.5 are given in [5].
**Definition 1.2.** An orbit for a set-valued mapping $F : X \to CB(X)$ at a point $x_0$ is a sequence $(x_n)$, where $x_n \in Fx_{n-1}$ for all $n$.

**Definition 1.3.** For two set-valued mappings $S$ and $T : X \to CB(X)$, we define an orbit at a point $x_0 \in X$, if there exists a sequence $(x_n)$ where $x_n \in Sx_{n-1}$ or $x_n \in Tx_{n-1}$ depending on whether $n$ is even or odd.

**Definition 1.4.** The metric space $X$ is said to be $x_0$-jointly orbitally complete, if every Cauchy sequence of each orbit at $x_0$ is convergent in $X$.

**Definition 1.5.** Let $F : X \to CB(X)$ be continuous. Then the mapping $x \to d(x, Fx)$ is continuous for all $x \in X$.

**Definition 1.6** [11]. If $A, B \in C(X)$ then for all $a \in A$, there exists a point $b \in B$ such that $d(a, b) \leq H(A, B)$.

**Lemma 1.7** [8]. Suppose that $\phi$ is a mapping of $[0, \infty)$ into itself, which is nondecreasing, upper-semicontinuous and $\phi(t) < t$ for all $\phi(t) > 0$. Then $\lim_{n \to \infty} \phi^n(t) = 0$, where $\phi^n$ is the composition of $\phi$ $n$ times.

2. Main result

**Theorem 2.1.** Let $S$ and $T$ be mappings of a metric space $X$ into $C(X)$ and let $X$ be $x_0$-jointly orbitally complete for some $x_0 \in X$. Suppose that $p > 0$ and for all $x, y \in X$ satisfying:

$$H^p(Sx, Ty) \leq \phi(adp(x, y) + (1 - a) \max \{D^p(x, Sx), D^p(y, Ty)\}),$$  \hspace{1cm} (2.1)

where $a \in (0, 1)$ and $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing, upper-semicontinuous and $\phi(t) < t$ for all $t > 0$. Then $S$ and $T$ have a common fixed point in $X$.

**Proof.** Let $x_0 \in X$. For any $x_1 \in Sx_0$, then by Definition 1.6, there exists a point $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq H(Sx_0, Tx_1)$. The choice of the sequence $(x_n)$ in $X$ guarantees that

$$x_n \in Sx_{n-1} \text{ if } n \text{ is even, } x_n \in Tx_{n-1} \text{ if } n \text{ is odd.}$$  \hspace{1cm} (2.2)

Now, we claim that $d(x_1, x_2) \leq d(x_0, x_1)$. Suppose $d(x_1, x_2) > d(x_0, x_1)$ and $\varepsilon = d(x_1, x_2)$. Then by using (2.1) it follows that

$$\varepsilon = d(x_1, x_2) \leq H(Sx_0, Tx_1)$$  

$$\leq [\phi(adp(x_0, x_1) + (1 - a) \max \{D^p(x_0, Sx_0), D^p(x_1, Tx_1)\})]^{1/p}$$  

$$\leq [\phi(a\varepsilon^p + (1 - a)\varepsilon^p)]^{1/p}$$  

$$\leq [\phi(\varepsilon^p)]^{1/p} < \varepsilon, \text{ a contradiction.}$$  \hspace{1cm} (2.3)

Therefore $d(x_1, x_2) \leq d(x_0, x_1)$ and

$$d^p(x_1, x_2) \leq H^p(Sx_0, Tx_1)$$  

$$\leq \phi(adp(x_0, x_1) + (1 - a) \max \{D^p(x_0, Sx_0), D^p(x_1, Tx_1)\})$$  \hspace{1cm} (2.4)

$$\leq \phi(\varepsilon^p(x_0, x_1)).$$
Similarly, we have \( d^n(x_2, x_3) \leq \phi(d^n(x_1, x_2)) \leq \phi^2(d^n(x_0, x_1)) \).

Proceeding in this way, we have

\[
d^p(x_n, x_{n+1}) \leq \phi^n(d^p(x_0, x_1)) \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.5}
\]

By Lemma 1.7, it follows that

\[
\lim_{n \to \infty} d^p(x_n, x_{n+1}) = 0, \quad \text{that is, } \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.6}
\]

In order to prove that \( \{x_n\} \) is a Cauchy sequence, it is sufficient to show that \( \{x_{2n}\} \) is a Cauchy sequence. Suppose that \( \{x_{2n}\} \) is not a Cauchy sequence. Then there is an \( \varepsilon > 0 \) such that for a sequence of even integers \( \{n(k)\} \) defined inductively with \( n(1) = 2 \) and \( n(k+1) \) is the smallest even integer greater than \( n(k) \) such that

\[
d(x_{n(k+1)}, x_{n(k)}) > \varepsilon. \tag{2.7}
\]

So that

\[
d(x_{n(k+1)-2}, x_{n(k)}) \leq \varepsilon. \tag{2.8}
\]

It follows that

\[
\varepsilon < d(x_{n(k+1)}, x_{n(k)}) \leq d(x_{n(k+1)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)-2}) + d(x_{n(k)-2}, x_{n(k)}) \tag{2.9}
\]

for \( k = 1, 2, 3, \ldots \). Using (2.6) and (2.8) it follows that

\[
\lim_{k \to \infty} d(x_{n(k+1)}, x_{n(k)}) = \varepsilon. \tag{2.10}
\]

By the triangle inequality, we have

\[
|d(x_{n(k+1)}, x_{n(k)}) - d(x_{n(k)}, x_{n(k)-1})| \leq d(x_{n(k+1)}, x_{n(k)-1}),
\]

\[
|d(x_{n(k)-1}, x_{n(k)+1}) - d(x_{n(k+1)}, x_{n(k)})| \leq d(x_{n(k+1)}, x_{n(k)-1}). \tag{2.11}
\]

It follows from (2.6) and (2.10) that

\[
\lim_{k \to \infty} d(x_{n(k)}, x_{n(k)-1}) = \lim_{k \to \infty} d(x_{n(k+1)-1}, x_{n(k)+1}) = \varepsilon. \tag{2.12}
\]

Using (2.6), we have

\[
D(x_{n(k+1)}, x_{n(k)}) \leq d(x_{n(k+1)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})
\]

\[
\leq H(Sx_{n(k+1)-1}, Tx_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}). \tag{2.13}
\]

and using (2.1), we have

\[
H^p(Sx_{n(k+1)-1}, Tx_{n(k)})
\]

\[
\leq \phi(ad^p(x_{n(k+1)-1}, x_{n(k)})) + (1-a) \max \{D^p(x_{n(k)+1}, x_{n(k+1)-1}), D^p(x_{n(k)}, Tx_{n(k)})\}. \tag{2.14}
\]

Using (2.8), (2.10), (2.13), (2.14), and upper semi-continuity of \( \phi \) it follows by letting \( k \to \infty \) that

\[
\varepsilon \leq [\phi(a\varepsilon^p)]^{1/p} \leq [\phi(\varepsilon^p)]^{1/p} < \varepsilon, \tag{2.15}
\]
a contradiction. Therefore, \( \{x_{2n}\} \) is a Cauchy sequence in \( X \) and since \( X \) is \( x_0 \)-jointly orbitally complete metric space, so the sequence \( \{x_n\} \) of each orbit at \( x_0 \) is convergent in \( X \). Therefore there exists a point \( z \in X \) such that \( x_0 \to z \).

Then again using (2.1), we have

\[
D^p(x_{2n-1}, Tz) \leq H^p(Sx_{2n-2}, Tz) \\
\leq \phi(ad^p(x_{2n-2}, z) + (1-a) \max \{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\})
\]

or equivalent to

\[
D^p(x_{2n-1}, Tz) \leq \phi(ad^p(x_{2n-2}, z) + (1-a) \max \{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\}).
\]

(2.16)

Now taking \( n \to \infty \) in (2.17), then we have \( D^p(z, Tz) \leq \phi((1-a)D^p(z, Tz)) \) if \( z \notin Tz \), a contradiction. Thus \( z \in Tz \).

Similarly, we show that \( z \in Sz \). Hence, \( z \in Sz \cap Tz \). This completes the proof.

\[\Box\]

**Open problem.** What further restrictions are necessary for the convergence of the sequence \( \{x_n\} \) if \( \phi \) is dropped from (2.1)?

**Acknowledgement.** The authors would like to express their deep gratitude to the referee for helpful comments and suggestions to present the original paper into this form.

**References**


M. R. Singh: Department of Mathematics, Manipur University, Canchipur, Imphal, 795 003, Manipur, India

L. S. Singh: Department of Mathematics, D.M. College of Science, Imphal, 795 001, Manipur, India

P. P. Murthy: Department of Mathematics, Arignar Anna Government Arts College, Karaikal, Pondichery, U.T.-609605, India