REGULAR-UNIFORM CONVERGENCE AND THE OPEN-OPEN TOPOLOGY

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Abstract. In 1994, Bânzaru introduced the concept of regular-uniform, or \( r \)-uniform, convergence on a family of functions. We discuss the relationship between this topology and the open-open topology, which was described in 1993 by Porter, on various collections of functions.

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1. Introduction. In [1], Bânzaru introduced the concept of regular-uniform, or \( r \)-uniform, convergence on a family of functions \( F \subset Y^X \) and proved a number of facts about the topological space \( (F, T_r) \) where \( T_r \) is the topology induced by this convergence. Porter introduced the open-open topology [5] in 1993 and proved that on families of self-homeomorphisms on \( X \) that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence [3]; this in fact is true on \( C(X, Y) \), the collection of all continuous functions from \( X \) to \( Y \). We shall show that the topology of \( r \)-uniform convergence on any subfamily \( F \) of the class of all continuous functions on \( X \) into \( Y \) is equivalent to the open-open topology [5], \( T_{oo} \), on \( F \) and hence, equivalent to the topology of Pervin quasi-uniform convergence on \( F \).

Throughout this paper let \( (X, T) \) and \( (Y, T') \) be topological spaces. We will use \( Y^X \) to mean the collection of all functions from \( X \) into \( Y \) while \( C(X, Y) \) will represent the collection of all continuous functions from \( X \) into \( Y \), and \( H(X) \) is the collection of all self-homeomorphisms on \( X \).

2. Preliminaries. A net of functions \( \{f_\alpha: (X, T) \to (Y, T')\}_{\alpha \in I} \) converges \( r \)-uniformly (or regular uniformly) to \( f \in Y^X \) [1] if and only if for any \( O \in T' \) such that \( f^{-1}(O) \neq \emptyset \), there exists \( i_0 \in I = [0, 1] \) such that \( f_i(x) \in O \) for all \( i \geq i_0 \) and for all \( x \in f^{-1}(O) \). This convergence defines a topology on \( F \) called the topology of \( r \)-uniform or regular uniform convergence.

In the same paper, Bânzaru also defined a topology, \( T_r \), on \( F \subset Y^X \) as follows: let \( f \in F \) and \( O \in T' \). Set

\[
S(f; O) = \{g \in F : g(f^{-1}(O)) \subset O\},
\]

then \( S = \{S(f; O) : f \in F \text{ and } O \in T'\} \) is a subbasis for a topology \( T_r \) on \( F \). Bânzaru then proved that this topology \( T_r \) on \( F \) is actually equivalent to the topology of \( r \)-uniform convergence on \( F \).
Now let $O \in T$ and $U \in T'$ and define
\[(U,V) = \{h \in F : h(O) \subset U\}. \tag{2.2}\]

Then $S_{oo} = \{(O,U) : O \in T \text{ and } U \in T'\}$ is a subbasis for the open-open topology, $T_{oo}$, \cite{5} on $F$.

In addition, the set $S_{co} = \{(C,U) \subset F : C \text{ is compact in } X \text{ and } U \text{ is open in } Y\}$ is a subbasis for the well-known compact-open topology, $T_{co}$, on $F$.

Let $X$ be a nonempty set and let $Q$ be a collection of subsets of $X \times X$ such that
1. for all $U \in Q$, $\triangle = \{(x,x) \in X \times X : x \in X\} \subset U$,
2. for all $U \in Q$, if $U \subset V$ then $V \in Q$,
3. for all $U,V \in Q$, $U \cap V \in Q$, and
4. for all $U \in Q$, there exists some $W \in Q$ such that $W \circ W \subset U$ where $W \circ W = \{(p,q) \in X \times X : \text{there exists some } r \in X \text{ with } (p,r),(r,q) \in W\}$ then $Q$ is a quasi-uniformity on $X$.

A quasi-uniformity, $Q$, on $X$ induces a topology, $T_{Q}$, on $X$, where for each $x \in X$, the set $\{U[x] : U \in Q\}$ is a neighborhood system at $x$ where $U[x]$ is defined by $U[x] = \{y \in X : (x,y) \in U\}$.

A family, $S$ of subsets of $X \times X$ which satisfies
(i) for all $R \in S$, $\triangle \subset R$, and
(ii) for all $R \in S$, there exists $T \in S$ such that $T \circ T \subset R$, is a subbasis for a quasi-uniformity, $Q$, on $X$. This subbasis $S$ generates a basis, $B$, for the quasi-uniformity, $Q$, where $B$ is the collection of all finite intersections of elements of $S$. The basis, $B$, generates the quasi-uniformity $Q = \{U \subset X \times X : \hat{B} \subset U \text{ for some } \hat{B} \in B\}$.

For a more thorough background on quasi-uniform spaces, see \cite{2}.

In 1962, Pervin \cite{4} constructed a specific quasi-uniformity which induces a compatible topology for a given topological space. His construction is as follows: Let $(X,T)$ be a topological space. For $O \in T$ define
\[S_{o} = (O \times O) \cup ((X \setminus O) \times X). \tag{2.3}\]

One can show that for $O \in T$, $S_{o} \circ S_{o} = S_{o}$ and $\triangle \subset S_{o}$, hence, the collection $\{S_{o} : O \in T\}$ is a subbasis for a quasi-uniformity, $P$, on $X$, called the Pervin quasi-uniformity.

Let $Q$ be a compatible quasi-uniformity for $(X,T)$ and let $F \subset C(X,Y)$. For $U \in Q$, define the set
\[W(U) = \{(f,g) \in F \times F : (f(x),g(x)) \in U \text{ for all } x \in X\}. \tag{2.4}\]

Then the collection $B = \{W(U) : U \in Q\}$ is a basis for a quasi-uniformity, $Q^{*}$, on $F$, called the quasi-uniformity of quasi-uniform convergence with respect to $Q$ \cite{3}. The topology, $T_{Q^{*}}$, induced by $Q^{*}$ on $F$, is called the topology of quasi-uniform convergence with respect to $Q$. If $Q$ is the Pervin quasi-uniformity, $P$, then $T_{P^{*}}$ is called the topology of Pervin quasi-uniform convergence.
3. The topologies. We first extend, to subsets of $C(X,Y)$, the result from [5] that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence on a subgroup $G$ of $H(X)$.

**Theorem 3.1.** Let $F \subset C(X,Y)$. The open-open topology, $T_{oo}$, is equivalent to the topology of Pervin quasi-uniform convergence, $T_{ps}$, on $F$.

**Proof.** Assume $F \subset C(X,Y)$. Let $(O,U)$ be a subbasic open set in $T_{oo}$ and let $f \in F$. Then $f(O) \subset U$. So $f \in W(S_U)\{f\}$ where

$$W(S_U)\{f\} = \{g \in F : (f(x),g(x)) \in S_U = U \times U \cup (X \setminus U) \times X, \forall x \in X\}. \quad (3.1)$$

Hence, if $g \in W(S_U)\{f\}$ and $x \in O$, then $f(x) \in U$ so $g(x) \in U$. Thus, $g \in (O,U)$ and $W(S_U)\{f\} \subset (O,U)$. Therefore, $T_{oo} \subset T_{ps}$.

Now let $V \in T_{ps}$ and $f \in F$. Then there exists $U \in P$ such that $f \in W(U)\{f\} \subset V$. Since $U \in P$, there exists some finite collection, $\{U_i : i = 1,2,\ldots,n\} \subset T$ such that $\cap_{i=1}^n S_{U_i} \subset U$. Define $A = \cap_{i=1}^n (f^{-1}(U_i),U_i)$. Then $A$ is an open set in $T_{oo}$ and $f \in A$. Assume $g \in A$ and let $x \in X$. If $f(x) \in U_j$ for some $j \in \{1,2,\ldots,n\}$, then $x \in f^{-1}(U_j)$ and $g \in A$, hence, $(f(x),g(x)) \in U_j \times U_j \subset S_{U_j}$. Therefore, $g \in W(\cap_{i=1}^n S_{U_i})\{f\} \subset W(U)\{f\} \subset V$ so that $A \subset V$. Therefore, $T_{oo} = T_{ps}$ on $F$. \hfill \square

Next we show that the regular-uniform topology is equivalent to the open-open topology on any subset, $F$, of $C(X,Y)$, and hence, also to the topology of Pervin quasi-uniform convergence on $F$.

**Theorem 3.2.** For $F \subset C(X,Y)$, $T_{oo} = T_r$ on $F$.

**Proof.** Note that a subbasic open set in $T_r$, $S(f;O) = \{g \in F : g(f^{-1}(O)) \subset O\}$ is equal to $(f^{-1}(O),O)$. Hence, if $f^{-1}(O)$ is open in $X$, which is the case when $f$ is continuous, $S(f;O)$ is a subbasic open set in $T_{oo}$. Therefore, $T_r \subset T_{oo}$.

Now let $(O,U)$ be a subbasic open set in $T_{oo}$ and let $f \in (O,U)$. Then $f(O) \subset U$ which implies that $O \subset f^{-1} \circ f(O) \subset f^{-1}(U)$. Since $f \circ f^{-1}(U) = U$, $f \in (f^{-1}(U),U) = S(f;U) \subset T_r$. If $g \in (f^{-1}(U),U)$, then $g(f^{-1}(U)) \subset U$. If $x \in O$, then $x \in f^{-1}(U)$ so that $g(x) \in U$ giving us that $g \in (O,U)$, whence $T_{oo} \subset T_r$ and we are done. \hfill \square

While it is always true that $T_{oo} \subset T_r$ on $F \subset Y^X$, it is not necessarily true that $T_r = T_{oo}$ for $F \subset Y^X$ as the following example shows.

**Example 3.3.** Define the sets $X = \{1,2,3\}$, $T = \{\{1\},\phi,X\}$, $Y = \{1,2,3,4\}$, $T' = \{\{1,2\},\{3,4\},\phi,Y\}$ and $F = \{f_1,f_2,f_3,f_4\}$ which are given in Table 3.1. Then $T_{oo} = \{\phi,F,\{f_1,f_2,f_3\},\{f_4\}\}$. But $S(f_3;\{3,4\}) = \{f_3\} \notin T_{oo}$. In fact, $T_r$ is the discrete topology on $F$.

Bânăşaru proved that for any $F \subset Y^X$, the compact-open topology, $T_{co}$, is coarser than $T_r$. However, although $T_{co} \subset T_{oo}$ on $F$ when $F \subset C(X,Y)$, it is not necessarily true that $T_{co} \subset T_{oo}$ for $F \subset Y^X$. Consider Example 3.3 again. We have that $\{\{2\},\{3,4\}\}$ is in $T_{oo}$ and equals $\{f_3\}$, but $\{f_3\} \notin T_{oo}$. In this example, the compact-open topology on $F$ is also the discrete topology and thus equals the regular-uniform topology on $F$. 
Another fact that has been proved in [1] about the regular-uniform topology is that if the topology for $Y$ is regular, then $(C(X,Y), T_r)$ is closed in $(Y^X, T_r)$. However, this is not true when $Y^X$ is given the open-open topology; that is, let $(X,T)$ and $(Y,T')$ be topological spaces such that $(Y,T')$ is regular. Then $(C(X,Y), T_r)$, which is the same as $(C(X,Y), T_{oo})$ is not necessarily closed in $(Y^X, T_{oo})$. The following example illustrates this.

**Example 3.4.** Let $X = \{1,2\}$, $T = \{\phi, X, \{1\}\}$, $Y = \{1,2,3\}$, and $T' = \{\phi, Y, \{1\}, \{2,3\}\}$. The collection $Y^X$ is given in Table 3.2. Note that $T'$ is a partition topology and is thus regular. Also note that $f_1^{-1}(\{2,3\}) = \{2\}$ and so $f_1$ is not continuous. The only open sets in $(Y^X, T_{oo})$ that contain $f_1$ are $(\phi, Y) = Y^X$ and $(\{1\}, \{1\}) = \{f_1, f_2, f_3\}$. Both of these sets contain the function $f_2$ which is continuous. Thus, $C(X,Y)$ is not closed in $(Y^X, T_{oo})$, even though $(Y, T')$ is regular.

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**References**


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